

Compressive Sensing for Ultrasound RF Echoes Using α -Stable Distributions

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Abstract—This paper introduces a novel framework for compressive sensing of biomedical ultrasonic signals based on modelling data with stable distributions. We propose an approach to ℓ_p norm minimisation that employs the iteratively reweighted least squares (IRLS) algorithm but in which the parameter p is judiciously chosen by relating it to the characteristic exponent of the underlying alpha-stable distributed data. Our results show that the proposed algorithm, which we prefer to call $S\alpha S$ -IRLS, outperforms previously proposed ℓ_1 minimisation algorithms, such as basis pursuit or orthogonal matching pursuit, both visually and in terms of PSNR.

I. INTRODUCTION

Ultrasound imaging is arguably the most widely used cross-sectional medical imaging modality worldwide. Indeed, ultrasound has a number of potential advantages over other medical imaging modalities, because it is non-invasive, portable and versatile, it does not use ionizing radiation, and it is relatively low-cost.

The general principle of ultrasound image formation involves the transmission of a train of pulses from an array of transducers (the probe) towards the plane being scanned. The returning echoes are then analysed in order to construct an image that displays their location and amplitude. Image compression is needed in order to reduce the data volume and to achieve a low bit rate, ideally without any perceived loss of image quality. The need for transmission bandwidth and storage space in the digital radiology environment, especially in telemedicine applications, and the continuous diversification of ultrasound applications keep placing new demands on the capabilities of existing systems. Introduction of new technologies, such as 4D ultrasound, potentially entailing orders of magnitude greater requirements for data transfer, processing, and storage, impose even greater demands and act to encourage the development of effective data reduction techniques.

In this paper, we propose to exploit the fairly new theory of compressive sensing (CS) [1], [2] to provide optimised ultrasound imaging solutions, while also having the potential to offer a coherent framework for image analysis, object detection, and information extraction. The contributions of

this paper are manifold but can be summarized in the following points: (i) we introduce a novel framework for ultrasound imaging based on CS; (ii) we propose an approach to ultrasound RF echoes reconstruction based on ℓ_p norm minimisation; (iii) we propose a principled way of choosing the parameter p in the reconstruction algorithm by modelling the RF echoes with alpha-stable distributions.

II. THEORETICAL PRELIMINARIES

Compressive sensing is based on measuring a significantly reduced number of samples than what is dictated by the Nyquist theorem. This has potential benefits in ultrasound imaging since it can facilitate reduced storage space and transmission bandwidth due to the inherent compression achieved. In the following two subsections, we provide a brief, necessary overview of the emerging theory of compressive sensing and a description of the heavy-tailed alpha-stable family of distributions, which we employ for ultrasound data modelling.

A. Compressive Sensing

Given a correlated image, the traditional transform-based compression method performs the following steps: i) acquires all N samples of the signal, (ii) computes a complete set of transform coefficients (e.g., DCT or wavelet), (iii) selectively quantises and encodes only the $K \ll N$ most significant coefficients. This procedure is inefficient, since a significant proportion of the output of the analogue-to-digital conversion process ends up being discarded.

Compressive Sensing is concerned with sampling signals more parsimoniously, acquiring only the relevant signal information, rather than sampling followed by compression. The main hallmark of this methodology is that, given a compressible signal, a small number of linear projections, directly acquired before sampling, contain sufficient information to effectively perform the processing of interest (signal reconstruction, detection, classification, etc).

In terms of signal approximation, Candès *et. al* [1] and Donoho [2] have demonstrated that if a signal is K -sparse in one basis (meaning that the signal is exactly or approximately represented by K elements of this basis), then it can be recovered from $M = c.K \ll N$ fixed (non-adaptive) linear projections onto a second basis, called the measurement basis, which is incoherent with the sparsity basis, and where $c > 1$ is a small overmeasuring constant. The measurement model is

$$y = \Phi x, \quad (1)$$

This work was partially funded by the Marie Curie TOK-DEV “ASPIRE” grant (MTKD-CT-2005-029791) within the 6th European Community Framework Program

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where x is the $N \times 1$ discrete-time signal, y is the $M \times 1$ vector containing the compressive measurements, and Φ is the $M \times N$ measurement matrix. Using the M measurements in the first basis and given the K -sparsity property in the other basis, the original signal can be recovered by taking a number of different approaches. The majority of these approaches solve constrained optimization problems. Commonly used approaches are based on convex relaxation (Basis Pursuit [1]), non-convex optimisation (Re-weighted ℓ_p minimisation [3]) or greedy strategies (Orthogonal Matching Pursuit (OMP) [4]).

B. Alpha-stable distributions for modelling ultrasound data

The ultrasound image formation theory has been long time dominated by the assumption of Gaussianity for the return RF echoes. However, the authors in [5] have shown that ultrasound RF echoes can be accurately modelled using a power-law shot noise model, which in [6] has been in turn shown to be related to alpha-stable distributions.

By definition, a random variable is called symmetric α -stable ($S\alpha S$) if its characteristic function is of the form:

$$\varphi(\omega) = \exp(j\delta\omega - \gamma|\omega|^\alpha), \quad (2)$$

where α is the *characteristic exponent*, taking values $0 < \alpha \leq 2$, δ ($-\infty < \delta < \infty$) is the *location parameter*, and γ ($\gamma > 0$) is the *dispersion* of the distribution. For values of α in the interval $(1, 2]$, the location parameter δ corresponds to the mean of the $S\alpha S$ distribution, while for $0 < \alpha \leq 1$, δ corresponds to its median. The dispersion parameter γ determines the spread of the distribution around its location parameter δ , similar to the variance of the Gaussian distribution.

Kutay *et al.* [5] have also shown that by assuming a $S\alpha S$ model for the RF echoes, the power spectrum of the envelope takes a power-law form as well. The corresponding density (sometimes referred to as generalised Rayleigh) is given in the following equation and parameter estimation approaches were proposed in [7].

$$p(x) = x \int_0^\infty u \exp(-\gamma u^\alpha) J_0(ux) du \quad (3)$$

where J_0 is the zeroth order Bessel function of the first kind.

1) *Log-moment estimation of model parameters*: Based on Mellin Transform [8], Nicolas [9] proposed the second-kind statistics theory, by analogy with the way in which common statistics are deduced based on Fourier Transform. The corresponding Method of Log-Cumulants (MoLC) is based on equating sample log-cumulants to their theoretical counterparts for a particular model and then solving the resulting system, much in the same way as in the classical method of moments.

In particular, the Mellin transforms of the $S\alpha S$ and the generalised Rayleigh densities are given in (4) and (5) respectively. Interestingly, the expression for the Mellin transform of the $S\alpha S$ density is the same as that for its fractional lower order moments [10], by letting $s = p + 1$

$$\Phi_{S\alpha S}(s) = \frac{\gamma^{\frac{s-1}{\alpha}} 2^s \Gamma(\frac{s}{2}) \Gamma(-\frac{s-1}{\alpha})}{\alpha \sqrt{\pi} \Gamma(-\frac{s-1}{2})}, \quad (4)$$

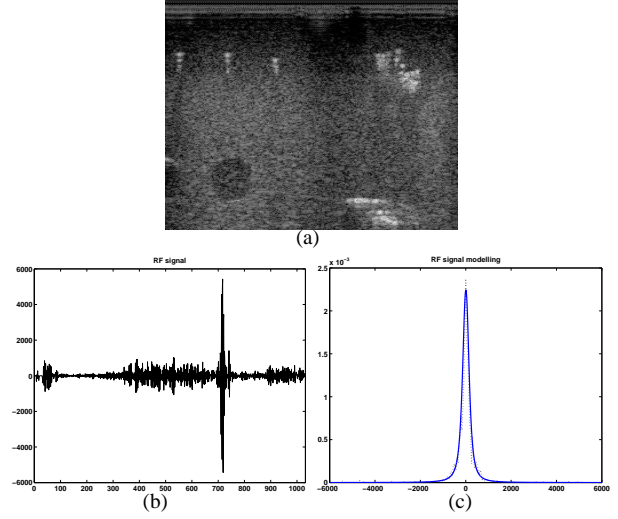


Fig. 1. Example RF signal modelling with $S\alpha S$ distributions. (a) Original (RF) ultrasound image. (b) An RF signal (one line of the original image). (c) $S\alpha S$ (continuous line) modelling of RF signal's histogram (dotted line).

$$\Phi_{gR}(s) = \frac{2^s \Gamma(\frac{s+1}{2}) \gamma^{\frac{s-1}{\alpha}} \Gamma(\frac{1-s}{\alpha})}{\Gamma(\frac{1-s}{2}) \alpha}. \quad (5)$$

By taking the limit as $s \rightarrow 1$ of the first and second derivatives of the logarithm of $\Phi_{S\alpha S}(s)$, we obtain the following results for the second-kind cumulants of the $S\alpha S$ model [11],

$$\begin{aligned} \tilde{k}_1 &= \frac{\alpha - 1}{\alpha} \psi(1) + \frac{\log \gamma}{\alpha} \\ \tilde{k}_2 &= \frac{\pi^2 \alpha^2 + 2}{12 \alpha^2} \end{aligned} \quad (6)$$

whereas proceeding the same way for $\Phi_{gR}(s)$ we obtain the following results for the second-kind cumulants of the generalised Rayleigh model [7],

$$\begin{aligned} \tilde{k}_1 &= -\psi(1) \frac{1-\alpha}{\alpha} + \log(2\gamma^{\frac{1}{\alpha}}) \\ \tilde{k}_2 &= \frac{\psi(1, 1)}{\alpha^2} \end{aligned} \quad (7)$$

where ψ is the Digamma function and $\psi(r, \cdot)$ is the Polygamma function, *i.e.*, the r -th derivative of the Digamma function. The first two sample second-kind cumulants can be estimated empirically from N samples y_i as follows:

$$\begin{aligned} \hat{k}_1 &= \frac{1}{N} \sum_{i=1}^N [\log(y_i)] \\ \hat{k}_2 &= \frac{1}{N} \sum_{i=1}^N [(\log(y_i) - \hat{k}_1)^2]. \end{aligned} \quad (8)$$

The estimation process simply involves solving either (6) or (7) for α and γ , depending on whether we want to model the RF image or the envelope detected one. In Fig. 1 we show an example of an ultrasound RF echo modelled using a $S\alpha S$ density function.

III. SIGNAL RECONSTRUCTION VIA ℓ_p NORM MINIMISATION

Ideally, our aim in this Section should be to reconstruct a vector x , the ultrasound echo in our case, with the smallest

number of non-zero components, that is, with the smallest ℓ_0 norm. Although the problem of finding such an x is NP-hard, there exist several sub-optimal strategies which are used in practice. Most of them solve a constrained optimization problem by employing ℓ_2 or ℓ_1 norms. On the other hand, CS reconstruction methods were developed in recent work (e.g., [3], [12]) by employing ℓ_p norms with $p < 1$, with the goal of approximating the ideal ℓ_0 case. Specifically, the problem consists in finding the vector x with the minimum ℓ_p norm by minimising

$$\hat{x} = \min \|x\|_p \quad \text{subject to} \quad \Phi x = y. \quad (9)$$

However, very few authors attempt to devise a principled strategy for choosing the optimal p or to relate the ℓ_p norm minimisation to the actual statistics of the signal to reconstruct. Indeed, for alpha-stable signals, which do not possess finite second- or higher-order moments, the minimum dispersion criterion [10], [13] can be defined as an alternative to the classical minimum mean square error for Gaussian signals. This leads naturally to a least ℓ_p norm estimation problem, an approach that can enhance the reconstruction of heavy-tailed signals from their measurement projections [2].

A method for choosing the optimum value of p has been proposed in [14], which was based on minimising the standard deviation of a FLOM-based covariation estimator. Other authors suggest the optimal p should be lower but as close as possible to the value of α , if α can be inferred. In our case, since the measurements y are merely linear combinations of the elements of x , by employing the stability property for stable random variable [10], we can use y to estimate the parameter α . In Section IV we compare both ways of choosing p .

Our approach to ℓ_p norm minimisation relies on the iteratively reweighted least squares approach (IRLS) [3] but is modified to incorporate the assumption of $S\alpha S$ distributed signals. The hallmark of the IRLS algorithm is to replace the ℓ_p objective function in (9) by a weighted ℓ_2 norm

$$\min \sum_{i=1}^N w_i x_i^2 \quad \text{subject to} \quad \Phi x = y \quad (10)$$

The whole algorithm is provided in pseudo-code below.

$S\alpha S$ -IRLS algorithm

- 1) Initialisation ($n = 0$): $\hat{x}^{(0)} = \min \|y - \Phi x\|^2$. Set the damping factor $\varepsilon = 1$.
- 2) Estimate α from y using either (6) or (7).
- 3) Determine p as a function of α either as in [14] or simply as $p = \alpha - 0.01$
- 4) **while** ε is greater than a pre-defined value **do**
 - a) $n = n + 1$
 - b) Find the weights $w_i = \left(\left(x_i^{(n-1)} \right)^2 + \varepsilon \right)^{\frac{p}{2}-1}$
 - c) Form a diagonal matrix Q_n whose entries are $\frac{1}{w_i}$
 - d) Form the solution to (10) as $\hat{x}^{(n)} = Q_n \Phi^T (\Phi Q_n \Phi^T)^{-1} y$

- e) **if** the norm of the residual, $\|\hat{x}^{(n)} - \hat{x}^{(n-1)}\|^2$, was reduced by a certain factor, **then** decrease ε .
- f) $\hat{x}^{(n-1)} = \hat{x}^{(n)}$

Note that the estimate of α above can be obtained using either (6) or (7) depending on whether we want to reconstruct the RF echoes or the envelope detected ultrasound image. In the case of RF echoes, utilising a maximum likelihood approach (e.g., [15]) is an equally valid option.

IV. SIMULATION RESULTS

The experiments were conducted using as source data one echo line arbitrarily chosen from each of six different ultrasound RF images of the same phantom. These data were then sampled using linear projections on random Gaussian bases at three levels. The three levels represent the number of samples taken from the original signal (the echo lines), these are 40, 50 and 60% (that is, $M = 0.4N$ etc.); hence concrete cases of undersampling the source are demonstrated.

Reconstruction of the samples was achieved by using the reweighted ℓ_p -norm minimization scheme (as described in [3]) for two values of p : p_{opt} and $p = \alpha - 0.01$, and for comparison, reconstructions using ℓ_1 -norm minimization with Basis Pursuit (BP) [1] and OMP [4] are shown (Table I). The values of α , and so p_{opt} (which is derived from α), were estimated directly from the ultrasound RF signal.

An analysis of the results was undertaken by calculating the PSNR of the reconstructed echo line compared with the original echo line. It can be seen in Table I that the best case is observed for reweighted ℓ_p -norm minimization with $p = \alpha - 0.01$, and similarly faithful results for p_{opt} . BP and OMP offer results below the level of reweighted ℓ_p -norm minimization, with OMP faring particularly badly by comparison. The results support the suggestion that reconstructing using a norm-minimizer with $p < 1$ produces better reconstructions of the signal in terms of the PSNR achieved. Taking into account prior statistical information of the signal is also shown to optimise reconstruction.

Table 2 shows the time taken during reconstruction for each of the methods employed in this simulation. Clearly, where the ℓ_p -norm minimization optimizes reconstruction in terms of PSNR, it currently gains no advantage in execution time.

For a qualitative analysis, patches from one of the ultrasound RF images were sampled and reconstructed, with $M = 0.6N$ measurements (Fig. 2). Visually, it can be seen that the OMP reconstruction introduces much distortion to the signal, and the $\ell_{p_{opt}}$ -norm reconstruction appears to distort in areas of high signal energy. The BP reconstruction shows greater noise than the $\ell_{\alpha-0.01}$ -norm reconstruction, the best compared to the original by eye and confirming the results indicated by the PSNR values obtained.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed and started to develop a novel framework for compressive ultrasound imaging. We have shown through simulations that RF echoes can be modeled

RF Image	% Samples	Method			
		$\ell_{p_{opt}}$	$\ell_{\alpha-0.01}$	BP	OMP
1	40	45.9	48.1	36.2	28.9
	50	47.4	49.2	38.2	32.9
	60	49.4	50.5	40.0	33.8
2	40	63.1	63.8	51.3	40.5
	50	64.7	65.4	54.6	42.8
	60	66.1	67.3	57.0	43.8
3	40	51.3	52.9	40.0	32.5
	50	51.9	54.4	43.2	34.5
	60	54.4	55.8	45.4	36.8
4	40	44.4	49.4	46.4	40.0
	50	49.0	52.3	50.6	40.7
	60	61.0	68.4	55.9	42.1
5	40	50.3	52.4	45.6	45.2
	50	51.3	53.0	46.1	46.3
	60	51.8	54.9	48.5	47.1
6	40	54.1	55.0	52.4	44.2
	50	56.6	60.6	56.0	45.3
	60	75.0	74.6	59.7	46.4

TABLE I

COMPARISON OF METHODS FOR RECONSTRUCTING RF ECHO LINES FROM 6 RF FRAMES WITH 40, 50 AND 60% OF SAMPLES FROM THE ORIGINAL. VALUES ARE PSNR (IN DB) COMPARING THE RECONSTRUCTION WITH THE ORIGINAL.

Method	% Samples		
	40	50	60
$\ell_{p_{opt}}$	305.2	471.2	651.0
$\ell_{\alpha-0.01}$	122.8	169.9	226.9
BP	3.66	3.60	3.73
OMP	1.74	2.51	3.52

TABLE II

EXECUTION TIMES (S) FOR RECONSTRUCTION WITH 40, 50 AND 60% OF SAMPLES FROM THE ORIGINAL.

using $S\alpha S$ densities, and the corresponding characteristic exponent α used to drive an ℓ_p -norm minimization problem, enhancing the reconstruction of the signal over existing ℓ_1 -norm minimization methods. Our current research focusses on developing algorithms and architectures for the actual acquisition of ultrasound images in a compressive fashion. Results will be reported in a future communication.

VI. ACKNOWLEDGMENTS

The authors would like to thank Dr. Haidong Liang from Bristol General Hospital for providing the ultrasound data used in this paper.

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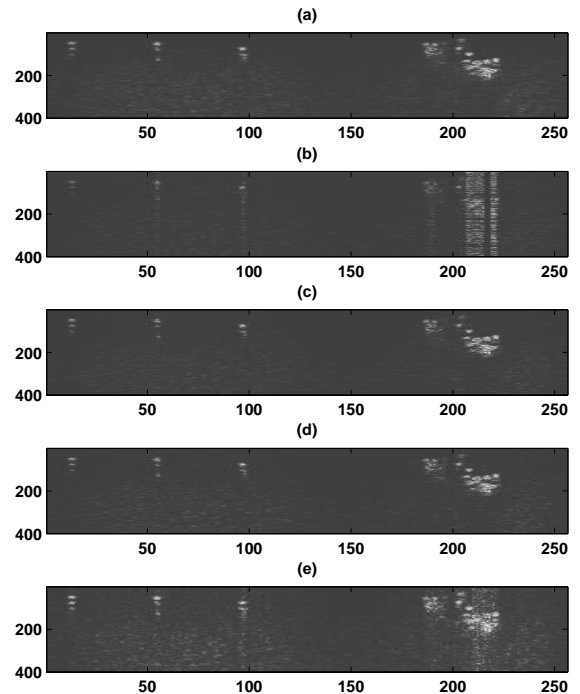


Fig. 2. Comparison of original and reconstructed patches. (a) Original (RF) ultrasound image. Reconstructions with (b) $\ell_{p_{opt}}$ -norm (c) $\ell_{\alpha-0.01}$ -norm (d) BP and (e) OMP.

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