# Bumps, blips and why do things localise?

The stock market crashes due to a single rogue trader. War starts from a lone sniper's bullet. The plane falls out the sky because of the tiniest crack or defect. Ships capsize after encountering a freak wave. So, in our increasingly safety-conscious and sanitised world, we should seek and eliminate such extreme events or imperfections, shouldn't we? Not so! argues Professor ***Alan Champneys***, Head of the Engineering Mathematics Department. Localised events do not always require localised causes.

Our story starts with the tale of two mathematicians at the end of the 19th century, Henri Poincaré (1854-1912) and Gusta Mittag-Leffler (1846-1927). For them, as for almost all scholars of dynamics in their day, the big motivating problem was the difficulty in reassuring the public that the moon and the planets were likely to evolve on their current course for the foreseeable future. Although today the media has now shifted our focus to possible fatal meteor strikes, for generation after generation we have looked to the heavens for portents of the end of the world.

Poincaré was the greatest mathematician of his day. A deep thinker who understood and made contributions to just about every branch of mathematics. Mittag-Leffler, by contrast, was a networker. He knew all the leading mathematicians of his day, corresponded with them, organised conferences, edited the key journal, and persuaded the King of Sweden to fund a unique prize. The winning entry would be the best solution to any one of a number of great unsolved mathematical problems of the day. Poincaré's essay attempted to show that the universe really does run like clockwork – and within the essay lay the foundations for two brand new fields of mathematics: topology, or 'rubber sheet geometry', and what is now popularly called 'chaos theory'. Needless to say, Poincaré won the competition.

Poincaré's essay studied a simplified model consisting of just three planets – for argument's sake, the sun, the moon and the Earth. He imagined the moon as a pawn in a great celestial chess game that could be positioned anywhere at will. Now, there are several positions in space where, if the moon were placed in any one of them, it would stay there for a very long time. These positions are called Lagrange points – locations in space where the gravitational forces acting on the moon are precisely counterbalanced by the orbital motion of the moon. Some of these points, such as its current position, are stable. Others are inherently unstable – like a pendulum standing on its head: given the minutest kick, the pendulum (moon) would fall to the left or right. Poincaré's idea was to study all the places where you could put the moon such that the path of its orbit would end up at a Lagrange point infinitely far in the future – the INset. Similarly, points that were found infinitely far in the past formed the OUTset. But what would happen if the INset and OUTset were to intersect? Continuing the pendulum analogy, there would be a motion that begins life upside down in the infinite past, gathers momentum and swings back up to the upright again, staying there for ever more. This 'homoclinic orbit', to give it its technical name, is described as ➜

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→ being 'localised in time'; it's just a blip, a finite lifetime of swing emerging from the upside down and eventually settling back upside down again, for all eternity.

Today we understand that homoclinic orbits form the backbone of chaotic dynamics. Their existence in any abstract set of mathematical equations – whether modelling the solar system, a pendulum, the stock market or whatever – can delineate the boundary between the regular, predictable and stable, and the chaotic, unpredictable and noisy. In essence, Poincaré had shown that celestial mechanics is inherently chaotic and his discovery is now heralded as the first example that qualitative thinking about dynamics is often more powerful than quantitative computation.

However, it is not widely known that in fact Poincaré missed the homoclinic orbit altogether in his original prize-winning essay. It was only while contemplating a referee's comment that he realised his mistake. He then paid Mittag-Leffler more than the prize money to have the original article destroyed and the new, much-celebrated version was published. We see in this story two truths we already know but don't like to admit: first, that science doesn't pay, and second, that negative referee comments can sometimes be a blessing.

The theory of homoclinic orbits has come a long way since that time and Champneys' ongoing work looks at deeper connections of the same theory. For example, water can support travelling structures (waves) that are localised in both time *and* space – so-called 'solitary waves' – isolated bumps of raised fluid that move with a well-defined speed. These were first described by John Scott Russell, who is also famous for his role in the demise of Isambard Kingdom Brunel, but who enters our story in 1834 as the man who first described such a solitary wave, caused by a boat crossing a narrow aqueduct.

Homoclinic connections do not end there. The same theory can also help explain things that 'localise in space'. For example, what happens if you twist a stretched elastic band? Try it. After a couple of twists, the band takes on a spiral shape like a stick of barley sugar (or helix). A few more twists and it starts to writhe and double up on itself. The trigger for this second form is that the spatially uniform helix must first 'localise' into a homoclinic orbit along the length of the band, in the sense that the amplitude of the helix gets larger in the middle, before the band suddenly jumps into the self-intersecting double helix (see right). Applications of these ideas abound: from laying undersea pipelines, where twisting of the cable can cause localised writhes and hence snagging points, through the mechanics of DNA – the double helix – to an ingenious explanation of how climbing plant tendrils gain their strength. So what does this theory tell us about localised events? Yes, certainly they sometimes have clearly defined localised causes. But, Champneys claims, there are many more instances of localised events – catastrophic solitary waves (tsunamis), the time at which the pendulum falls over, the point of snapping of an undersea cable – for which it is the system itself that shows the propensity to localise. The search for the precise imperfection that triggers the instance or point of localisation is, like the identification of scapegoats, often a futile waste of time. Instead, the message of chaos theory to many areas of applied research is that we need to invest our effort in a greater qualitative understanding of 'the system' – whatever that may be. ■

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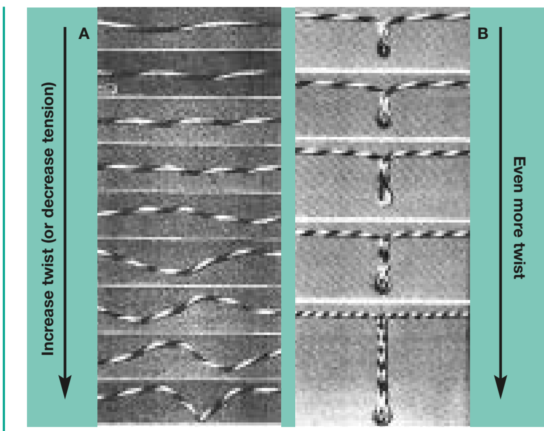


Twisting an elastic band