MATHEMATICAL PHYSICS/RANDOM MATRIX THEORY AND MATHEMATICAL STATISTICAL MECHANICS

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Feel free to contact me at thomas.bothner@bristol.ac.uk if you are interested in one of the projects below, I am happy to discuss specifics, prerequisites and learning outcomes. Under normal (non-COVID) circumstances you can also find me in my office 2A.15, Fry Building.

1. General overview

My research program is concerned with analytic and probabilistic questions in mathematical physics and I place particular emphasis on topics in random matrix theory which display intimate connections to mathematical statistical mechanics and the field of integrable differential equations. The application of asymptotic methods, special functions, probability theory, orthogonal polynomials and potential theory is central to this work. Please see http://orcid.org/0000-0001-8300-7467 for my complete publication list or https://sites.google.com/site/thomasbothner/extra-credit for a short summary of some of my published results

2. Current and future work

My current efforts lie at the forefront of research in mathematical statistical mechanics of highly correlated systems with focus on two major themes: exactly solvable lattice models and random matrices. The long-term goal is to unveil ground-breaking original connections between those themes and resolve a series of long-standing conjectures about the system's underlying analytic and asymptotic behaviors. The subsections below summarize two possible PhD projects in this area.

2.1. **From non-Hermitian to Hermitian random matrices.** This research project aims to investigate the remarkable connections between near Hermitian random matrices and the theory of integrable systems, with particular focus placed on near Hermitian random matrices with real entries. The concrete tasks attached to this largely unexplored direction consist in

(1) the exact calculation of eigenvalue gap and distribution functions of a near Hermitian random matrix with real entries, in the limit of large matrix sizes while the non-Hermiticity degree vanishes
(2) the rigorous study of the integro-differential dynamical systems that are expected to arise in the description of the same limiting gap and distribution functions

2.2. **Topological expansions for (near) random matrix models.** The planned work in this direction is concerned with the development of a method to derive large n asymptotics for the n-fold multiple integral

$$\mathcal{Z}_n[V,f] := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\det\big[f_j(x_k)\big]_{j,k=1}^{n}\prod_{1\leq j<k\leq n}(x_k-x_j)\prod_{k=1}^{n}\mathrm{e}^{-nV(x_k)}\mathrm{d}\nu(x_k) \qquad (1)$$

where $V:\mathbb{R}\to\mathbb{R}$ is a confining external field that grows sufficiently fast at infinity, $\{f_j\}_{j=1}^{n}$ a certain family of functions so that the multiple integral is well-defined and ν a (possibly discrete) measure on $\mathbb{R}$. The integrand above involves a Vandermonde two-body repulsive term plus the additional determinantal many-body interaction in terms of the family $\{f_j\}_{j=1}^{n}$. The main interest in the family (1) of n-fold multiple integrals stems from their occurence in the analysis of the six-vertex model with domain wall boundary conditions. Indeed, the computation of the limiting free energy in this model relies on the fact that its partition function is of the form (1) with $f_j(x)=x^j$. This (monomial) choice doubles the Vandermonde determinant in (1) and thus links $\mathcal{Z}_n[V,f]$ to a class of integrals frequently encountered in random matrix theory: with ν the Lebesgue measure, after integration over angular variables,

$$\mathcal{Z}_n[V,f]\propto_n\int_{\mathcal{H}_n}\mathrm{e}^{-n\,\mathrm{tr}\,V(M)}\,\mathrm{d}M =: \mathcal{Z}_n[V], \qquad (2)$$

where $\mathcal{H}_n$ is the space of $n\times n$ complex Hermitian matrices equipped with the Haar measure $\mathrm{d}M$. This matrix integral and certain generalizations thereof (integration in (2) then carried out over different groups) have been studied extensively for more than 20 years. The interest stems from the fact that such matrix integrals have combinatorial significance in that they count graphs embedded in surfaces according to their genus, precisely

$$\ln\left(\frac{\mathcal{Z}_n[V]}{\mathcal{Z}_n[V_q]}\right)\sim\sum_{g=0}^{\infty}\mathcal{F}_g n^{2-2g},\qquad V_q(x):=x^2 \qquad (3)$$

which holds for a class of polynomial V and where the coefficients $\mathcal{F}_g$ count the equivalence classes of graphs embedded in a certain compact, oriented surface of genus g. The asymptotic expansion (3) is commonly referred to as a topological expansion. On a technical level, the rigorous derivation of such a topological expansion rests on certain algebraic miracles which do not generalize to an arbitrary choice of $\{f_j\}_{j=1}^{n}$ in (1). This brings us to the following two tasks

(1) the derivation of topological expansions for the external source random matrix model (again relevant to the six-vertex model) with polynomial confining external field, $f_j(x)=\mathrm{e}^{a_jx}$ and both, discrete and continuous measures ν, and
(2) the development of a method to derive large n asymptotics for the integral (1) with more general functions $\{f_j\}_{j=1}^{n}$ in place.