Probability theory and analytic number theory

The Riemann zeta function is the unique meromorphic function ζ such that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for all s with $\Re(s) > 1$. This function has zeros at negative even integers, and the famous Riemann hypothesis states that all the other zeros (called non-trivial zeros) have a real part equal to $1/2$: this hypothesis has been checked for all zeros with imaginary part up to more than 10^{12}. At the beginning of the twentieth century, Hilbert and Pólya suggested the following spectral interpretation of the Riemann hypothesis: the non-trivial zeros of ζ may be the eigenvalues of $1/2 + iH$, where H is a self-adjoint operator. Such an operator has not been found so far.

However, Montgomery has proven and conjectued some statistical properties on the distribution of the zeros of ζ, and during his visit to the theoretical physicist Dyson in Princeton in 1972, they noticed that the properties of ζ proven by Montgomery are similar to those proven by Dyson on the distribution of the zeros of random Hermitian matrices. Since that visit, the connection between random matrix theory has been widely studied and developped.

From Hilbert and Pólya's perspective, it is natural to see the zeta function as some kind of characteristic polynomial of an operator, and then to compare ζ to characteristic polynomials of random matrices. From this point of view, a conjecture (still open) has been stated by Keating and Snaith in 2000: it is expected that for all positive integers k, the average of $|\zeta(1/2 + it)|^{2k}$ for t between 0 and T is equivalent, when $T \to \infty$, to $C_k(\log T)^{k^2}$ where $C_k > 0$ is an explicit constant. Keating and Snaith have proven a similar result for the characteristic polynomial of random unitary matrices (with the dimension N replacing $\log T$): they used this theorem in order to find the value of C_k in their conjecture on ζ.

A conjecture on extreme values of ζ (the maximum of $|\zeta(1/2 + i(t + h)|$ for $h \in [0, 1]$ and t uniformly distributed in $[0, T]$, $T \to \infty$) has been stated by Fyodorov, Hiary and Keating in 2012, together with a similar conjecture on the maximal modulus of the characteristic polynomial of random unitary matrices on the unit circle. These two conjectures have been much investigated in the last ten years and have been partially solved, in a number of papers by various authors (including Arguin, Belius, Bourgade, Chhaibi, Harper, Madaule, Najnudel, Paquette, Radziwill, Soundararajan, Zeitouni). A lot of different techniques have been used in the study of these conjectures, combining analytic number theory and probability theory for the conjecture on ζ. The study of extreme values of random branching processes is involved in a crucial way, and for the conjecture in random matrix theory, other tools are also used, including orthogonal polynomials on the unit circle and analysis of Riemann-Hilbert problems.

The goal of this project is to further explore links between random matrix theory and the Riemann zeta function, more generally, probability theory and analytic number theory. Here are some examples of mathematicals objects which can be studied in this project, besides those discussed above: random holomorphic functions, randomized modifications of ζ, random multiplicative functions on integers, coefficients of characteristic polynomials of random matrices.