

Some problems in geometric measure theory

I work in the area of geometric measure theory with a view to applications in analysis and partial differential equations. The below is a rather detailed description of a project which I would be interested to supervise.

Let A stand for a centred annulus in $\mathbb{R}^n$ for some $n \geq 2$ with inner radius $r > 0$ and outer radius $R > 0$. Let $\beta_{\text{in}}, \beta_{\text{out}} \in [-1, 1]$. Let E be a set contained in A with finite perimeter. A surface energy may be defined by

$$\mathcal{F}(E) := P(E; A) - \beta_{\text{in}}P(E; \mathbb{S}_r) - \beta_{\text{out}}P(E; \mathbb{S}_R).$$

Here, $P(E; A)$ corresponds to the perimeter of E inside the container A ; while $P(E; \mathbb{S}_r)$ corresponds to the surface area of the inner wall of the annulus *wetted* by E . The problem is to determine which sets E minimise the functional $\mathcal{F}(\cdot)$ when the volume $|E|$ of E is fixed. Physically, this corresponds to the equilibrium shape of a liquid drop contained in A in the absence of gravity when the container walls have adhesion coefficients $\beta_{\text{in}}, \beta_{\text{out}}$ (see [3] Chapter 19 and [1]). The case $\beta_{\text{in}} = \beta_{\text{out}} = 0$ corresponds to the relative isoperimetric problem while $\beta_{\text{in}} = \beta_{\text{out}} = -1$ corresponds to the constrained isoperimetric problem. The case when $\beta_{\text{in}}, \beta_{\text{out}}$ have equal magnitudes but opposite signs is of particular interest. The paper [2] solves the constrained isoperimetric problem in certain simply connected planar domains without necks (though the non-simply connected case is not discussed). The planar case $n = 2$ seems most amenable.

I might mention other problems too. For example, a version of the log-convex density conjecture but in higher codimension. This would involve working in the framework of integrable currents rather than sets of finite perimeter. Another area is the search for novel weighted Pólya-Szegő inequalities analogous to that associated with the Gaussian isoperimetric inequality but with non-Euclidean underlying geometry.

References

- [1] Finn, R., *Equilibrium capillary surfaces*, Springer-Verlag, 1986.
- [2] Leonardi, G.P., Saracco, G., *The isoperimetric problem in 2D domains without necks*, Calc. Var. 61, 56 (2022).
- [3] Maggi, F., *Sets of finite perimeter and geometric variational problems*, Cambridge University Press, 2012.