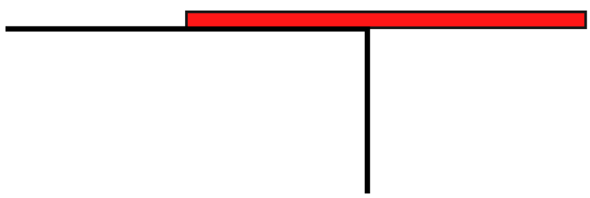
The one-sided arch 1

(by Alan Champneys)

Your task is to construct the longest possible overhanging arch to reach out over the sea, from a rigid clifftop, using only a very large collection of identical planks made of uniform material and length 2 metres. No nails, no glue, just planks, balanced one on top of the other.

We shall build up a solution bit by bit.

Using just one plank you can reach out precisely 1 metre with the plank teeting on the very edge of the cliff.



Can you think of a way of making a longer overhanging structure using two planks? What about three planks? or four? What is the longest overhang you can make?

Start drawing some pictures before doing any maths.

Solution

See the subsequent sheets for the solution that we have in mind. Of course, all kinds of other disigns with counterbalancing weights are also possible. These should be encouraged.

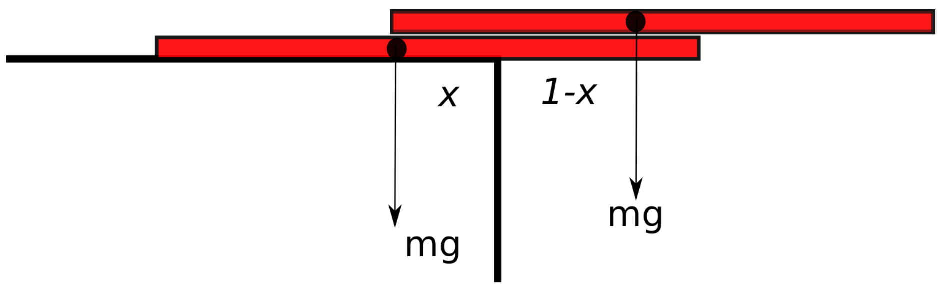
But the purpose of this exercise is to simply put one plank on top of any one plank.

The one-sided arch 2

(by Alan Champneys)

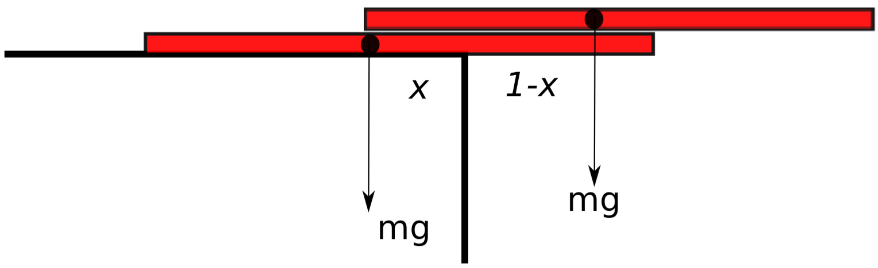
We are going to constrain things by saying that you can only place one plank on top of any other one and that all planks must point in the same direction.

Under this constraint, how far out can you place two planks, one on top of the other so that the top plank is teeting on the edge of bottom plank, and the bottom one is teeting on the edge of the cliff? *[The following diagram might help, where m is the mass of each plank]*



Solution

This can be solved either by taking moments or by trying to impose that the centre of mass should be at precisely the position of the pivot.



In either case, we find that a balance can only be obtained by setting that the mass times length on the left must equal to mass times length to the right of the pivot point. So, referring to the diagram.

$$mgx = mg(1-x), \qquad \Rightarrow 2x = 1 \quad \Rightarrow x = 1/2$$

So, the total overhang is

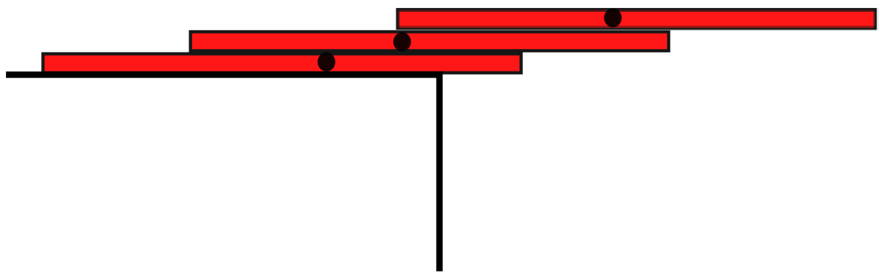
$$O = \text{overhang}_{\text{topplank}} + \text{overhang}_{\text{2ndplank}} = 1 + 1/2 = 3/2 \text{ metres}$$

The one-sided arch 3

(by Alan Champneys)

You should have found that if $x = 0.5$ metres then the two planks just teeter on the cliff edge, with a total overhang of $1 + x = 1.5$ metres.

Now we are going to proceed using the same approach, essentially taking the solution we already have for two planks, which we know is just teeters on the edge of the cliff, and add one more plank underneath, with the upper two just teetering on it. Take a look at the diagram

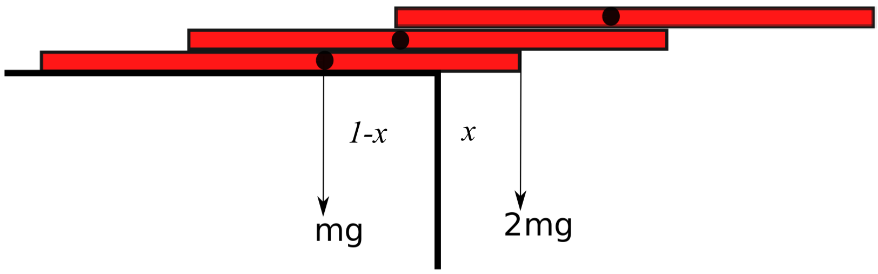


in which the centres of mass of each plank is marked by a large black blob.

Can you calculate the maximum overhang of the lower plank, so that the three planks just teeter on the edge of the cliff?

Solution

There are several ways to solve this, but the most efficient is to notice that we can treat the upper two planks as a single body with mass $2m$. Then the problem is just like the two-plank case, but with the upper plank having twice the mass.



Letting x be the additional overhang, balancing forces we get

$$2mgx = mg(1-x) \qquad \Rightarrow 3x = 1 \quad \Rightarrow x = 1/3$$

So

$$\text{Total overhang } O_3 = O_2 + 1/3 = 1 + 1/2 + 1/3$$

By now the smarter students may have begun to see where this is going ...

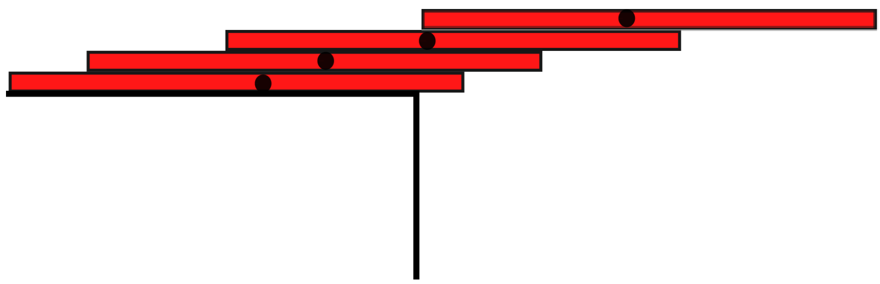
The one-sided arch 4

(by Alan Champneys)

You should have found the answer that the new piece of overhang is 1/3 metre, so that

$$\text{Total overhang} = 1 + 1/2 + 1/3 = 1.83333\,\text{metres}$$

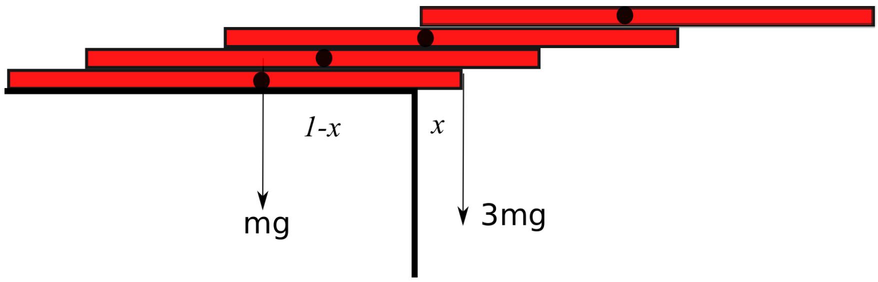
Now lets repeat this process by slotting a fourth plank underneath. Can you guess what the total overhang is now? How do you prove this?



Can you generalise this result to an arbitrary number of planks, N?

Solution

The case for four planks is similar,

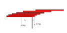


so the additional overhang x_4 is given by

$$3mgx_4 = mg(1 - x_4) \qquad \Rightarrow 4x_4 = 1 \quad \Rightarrow x_4 = 1/4$$

and

$$O_4 = O_3 + x_4 = 1 + 1/2 + 1/3 + 1/4.$$



Generalising to $n + 1$ planks, we get

$$x_{n+1} = 1/(n + 1), \qquad O_{n+1} = O_n + 1/(n + 1)$$

So, by induction

$$O_N = \sum_{n=1}^{N} \frac{1}{n}$$

The one-side arch 5

(by Alan Champneys)

So you should have found that with a total of N planks the overhang is equal to

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots + 1/N \text{ metres}$$

Note how the additional overhang gets smaller and smaller each time.

But what is the maximum length of overhang I can produce via this method if I keep taking more and more planks?

Mathematically, what we want to know is what is the sum as N tends to infinity of

$$\sum_{n=1}^{N} \frac{1}{n} = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots$$

This is known as the Harmonic Series. Try summing this on a calculator.

What do you think the limit is a $N \to \infty$?

How would you prove this?

Solution

This is a standard series. It tends to infinity. But very slowly. There are many ways to prove this and lots of possible extension ideas. See for example:

`https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)`