THE CENTRE FOR MARKET AND PUBLIC ORGANISATION

Seller Reputation and Trust in Pre-Trade Communication

Bruno Jullien and In-Uck Park

March 2011

Working Paper No. 11/272

Centre for Market and Public Organisation
Bristol Institute of Public Affairs
University of Bristol
2 Priory Road
Bristol BS8 1TX
http://www.bristol.ac.uk/cmpo/

Tel: (0117) 33 10952
Fax: (0117) 33 10705
E-mail: cmpo-admin@bristol.ac.uk

The Centre for Market and Public Organisation (CMPO) is a leading research centre, combining expertise in economics, geography and law. Our objective is to study the intersection between the public and private sectors of the economy, and in particular to understand the right way to organise and deliver public services. The Centre aims to develop research, contribute to the public debate and inform policy-making.

CMPO, now an ESRC Research Centre was established in 1998 with two large grants from The Leverhulme Trust. In 2004 we were awarded ESRC Research Centre status, and CMPO now combines core funding from both the ESRC and the Trust.

ISSN 1473-625X

University of Bristol

E·S·R·C ECONOMIC & SOCIAL RESEARCH COUNCIL

CMPO Working Paper Series No. 11/272

Seller Reputation and Trust in Pre-Trade Communication

Bruno Jullien[1]
and
In-Uck Park[2]

[1]*Toulouse School of Economics*
[2]*CMPO, University of Bristol*

March 2011

Abstract
We characterize the unique equilibrium in which high ability sellers always announce the quality of their items truthfully, in a repeated game model of experienced good markets with adverse selection on a seller's propensity to supply good quality items. In this equilibrium a seller's value function strictly increases in reputation and a seller's type is revealed within finite time. The analysis highlights a new reputation mechanism based on an endogenous complementarity the market places between a seller's honesty in pre-trade communication (trust) and his/her ability to deliver good quality (reputation). As maintaining honesty is less costly for high ability sellers who anticipate less "bad news" to disclose, they can signal their ability by communicating in a more trustworthy manner. Applying this model, we examine the extent to which consumer feedback systems foster trust in online markets, including the possibility that sellers may change identities or exit.

Keywords: cheap talk, consumer rating system, reputation, trust.

JEL Classification: C73, D82, D83, L14.

Electronic version: www.bristol.ac.uk/cmpo/publications/papers/2011/wp272.pdf

Acknowledgements
The authors thank Heski Bar-Isaac, Luis Cabral, Yuk-Fai Fong, David Levine, Thomas Mariotti, Jérome Mathis, Andrew McLennan, Fabien Postel-Vinay, Ron Siegel, Joel Sobel, Peter Sorensen, and Mike Whinston for useful discussion and comments, as well as seminar participants at Oxford University, the University Carlos III, and the Universities of Melbourne and Naples, and the CSIO/IDEI Workshop in Toulouse School of Economics.

Address for correspondence
CMPO, Bristol Institute of Public Affairs
University of Bristol
2 Priory Road
Bristol BS8 1TX
i.park@bristol.ac.uk
www.bristol.ac.uk/cmpo/

Seller Reputation and Trust in Pre-Trade Communication*

Bruno Jullien
Toulouse School of Economics
Toulouse, France

In-Uck Park
University of Bristol
Bristol, U.K.

March 26, 2011

Abstract. *We characterize the unique equilibrium in which high ability sellers always announce the quality of their items truthfully, in a repeated game model of experienced good markets with adverse selection on a seller's propensity to supply good quality items. In this equilibrium a seller's value function strictly increases in reputation and a seller's type is revealed within finite time. The analysis highlights a new reputation mechanism based on an endogenous complementarity the market places between a seller's honesty in pre-trade communication (trust) and his/her ability to deliver good quality (reputation). As maintaining honesty is less costly for high ability sellers who anticipate less "bad news" to disclose, they can signal their ability by communicating in a more trustworthy manner. Applying this model, we examine the extent to which consumer feedback systems foster trust in online markets, including the possibility that sellers may change identities or exit.* (*JEL* Codes: C73, D82, D83, L14)

Keywords: cheap talk, consumer rating system, reputation, trust.

1 Introduction

This paper characterizes the unique equilibrium in which high ability sellers always announce the quality of their items truthfully, in a repeated game model of experienced good markets with adverse selection on a seller's propensity to supply good quality items. In this equilibrium a seller's value function strictly increases in reputation and a seller's type is revealed within finite time. Applying this model, the paper examines the extent to which consumer feedback systems may foster trust in online markets, first when sellers cannot change identities or exit, and then when they can.

In online markets buyers cannot physically inspect the items for sale but only rely on the descriptions provided by the sellers regarding the quality of both the item itself and the delivery service. Since payment will already have been made when the buyer learns the quality (upon delivery), effective functioning of online markets hinges critically on the existence of a mechanism that warrants a sufficient level of trust amongst traders in

*The research is supported by the Leverhulme Trust, and was initiated while B. Jullien was visiting the University of Bristol. The authors thank Heski Bar-Isaac, Luis Cabral, Yuk-Fai Fong, David Levine, Thomas Mariotti, Jérome Mathis, Andrew McLennan, Fabien Postel-Vinay, Ron Siegel, Joel Sobel, Peter Sorensen, and Mike Whinston for useful discussion and comments, as well as seminar participants at Oxford University, the University Carlos III, and the Universities of Melbourne and Naples, and the CSIO/IDEI Workshop in Toulouse School of Economics. Emails: bjullien@cict.fr and i.park@bristol.ac.uk.

an ocean of strangers. Perhaps the most notable reaction of the market is the feedback systems widely adopted by online platforms, such as eBay, that allow buyers and sellers to leave publicly available comments about their trading partners.

Traditionally, word-of-mouth reputation systems have played an important role in experience good markets when legal systems of contract enforcement were absent or did not work effectively. What is attempted via feedback systems amounts to engineering this old wisdom in a new environment with certain special issues.[1] In a recent survey paper, Bajari and Hortacsu (2004) convey that existing empirical studies appear to point to the existence of a correlation between the "feedback score" of an online seller and the average quality delivered by that seller,[2] yet the primary causes for the observed correlation remain inconclusive.

This paper highlights a new reputation mechanism based on an endogenous complementarity the market places between a seller's honesty in pre-trade communication (trust) and his/her ability to deliver high quality goods (reputation). Reputation may be fostered because high ability sellers can better afford honest communication as they have less "bad news" to disclose, enhancing short-term communication (on the sale item's quality); At the same time, disparity in honesty across sellers of different ability accelerates long-term learning process (on the seller's true ability), allowing quicker reputation building. The reputational effect in our model is obtained via exploiting these reciprocal reinforcements between trust in communication and reputation in ability, which is a novel mechanism as far as we are aware. While this reputation mechanism has a particular pertinence to the role of online feedback systems mentioned above, the principal insight applies to experience good markets more generally.

Our approach stems from the recognition that there is a clear, albeit subtle, distinction between the information that is to be captured by the feedback scores of a seller, and that which is reflected in the prices that he fetches: The former is the level of "trust" the market places on the soft information provided by the seller prior to transaction concerning the quality, suitability, etc., of the particular item for trade; whereas the latter reflects the market's expectation of the quality of items delivered by him. Thus, the role of feedback systems is "to promote honest trade rather than to distinguish sellers who sell high quality products from those that sell low quality products" as pointed out by Dellarocas (2006).

Therefore, the observed correlation between the market's trust in a seller's honesty (feedback score) and its belief on his ability to deliver quality (price) is an indirect link that is yet to be accounted for. This paper presents a theoretical study that maps out the endogenous link between these two separate dimensions of "reputation," which we believe exists in marketplaces more broadly, beyond online markets.

[1] See Dellarocas (2003 and 2006) and references therein for a discussion of reputation issues on Internet, and Bar-Isaac and Tadelis (2008) for a survey of the literature on seller reputation.

[2] According to Bajari and Hortacsu (2004), a presence of 5-12% price premium for an "established" seller (with hundreds or thousands of mostly positively feedback scores) relative to a seller with no track record appears to be the most robust findings reported, e.g., by Melnik and Alm (2002), Livingston (2002), Kalyanam and McIntyre (2001), and Resnick et al. (2003). This positive effect of seller reputation on price is confirmed when more detailed data are used, by Houser and Wooders (2006) who control heterogeneous item descriptions and by Canals-Cerda (2008) who analyze panel (as opposed to cross-sectional) data.

Specifically, we study non-binding, pre-trade communication in buyer-seller relationships on a platform that maintains a full record of transaction history. First, we analyze a baseline case in which sellers have no place to trade other than a platform that can perfectly monitor sellers' identities and thus, prevent their history from being erased. We then extend the model to a full dynamic set-up in which sellers may restart with a clean record by obtaining a new identity, or may exit the platform for an outside option.

In our baseline model, a seller randomly draws an item of either good or bad quality in each period and announces this quality as cheap talk. Each seller is of one of two private types, high or low ability: a high-type seller draws a good quality item more frequently. Each item is traded at a price that is equal to the expected quality based on the market's belief regarding the seller's true ability, termed his "reputation," and his announcement. The buyer learns the true quality and reports it publicly, revealing the truthfulness of the seller's announcement (feedback system). The seller's reputation level is then updated accordingly. Here, we postulate fully reliable feedback comments because we are interested in the extent to which feedback mechanisms may help elicit trustworthy pre-trade communication from sellers.

As sellers' announcements are cheap talk, there always exists a "babbling" equilibrium in which announcements by sellers are completely ignored and thus, play no role at all. In this equilibrium, feedback comments induce only "pure learning" through simple observation of past quality, with no strategic effect on eliciting honest pre-trade communication.

As the polar benchmark, we focus on equilibria in which high-type sellers always announce truthfully, and we establish that there is a unique equilibrium of this kind, which we refer to as the "honest equilibrium".[3] In this equilibrium, each and every truthful announcement increases the seller's reputation, raising the price he receives in the next period if he claims his item to be of good quality. Low-type sellers falsely claim bad quality items to be good with a positive probability for short-term gain. A key intuition behind this result is that following a truthful announcement strategy is more costly for low-type sellers because they anticipate more "bad news" that they will have to disclose honestly. The probability of a low-type seller lying is a continuous but non-monotonic function of the prevailing reputation level, reaching one above a certain threshold reputation level.

Thus, communication facilitates information transmission in two separate dimensions. Firstly, it allows credible transmission of information about the item's quality, so that the price reflects the true quality of the good more accurately. Secondly, it speeds up the acquisition of information about the seller's type, because the truthfulness of the announcement, in addition to the delivered quality, reveals information on the seller's ability. Consequently, buyers learn the seller's ability more quickly, and more information is incorporated in the price of the item.

We also show that low-type sellers lie less frequently at all levels of reputation as they become more patient. This exerts diverging effects, however, on the two dimensions of information transmission. It clearly strengthens the informational contents of announcements on the item's quality, enhancing the credibility of the announcements and thereby, their impact on prices; but it reduces the informational content on the seller's type, atten-

[3] We adopt this terminology from Sobel (1985) described below.

uating the speed with which the market learns the seller's ability.

Then, we turn to the case in which sellers may start afresh at any time by obtaining a new identity after "milking" his reputation (in the same or another marketplace), which is a well-known problem with feedback systems.[4] To examine the extent to which such a possibility may undermine online reputation mechanisms, we extend the model to allow for entry and exit of sellers. We characterize a stationary equilibrium when sellers may at any time opt for an outside option of a fixed value, as well as when they can freely restart with a clean record in the same platform. We establish that in both cases a stationary equilibrium exists that exhibits all the aforementioned features of the honest equilibrium. However, we also show that reputation effects are weakened: Allowing such options increases cheating incentives by limiting the damage from abusing reputation and as a result, low-type sellers lie more frequently across all reputation levels than when such options were absent. As a consequence of reduced confidence, the price for good quality items is lower, but high-type sellers are able to build reputation at a faster speed since honest announcements of bad quality are received more favorably.

Allowing for entry and exit also brings extra insights as these decisions may convey some information about the type of the seller. For instance, Bar-Isaac (2003) presents a model similar to ours but with no scope for pre-trade communication (because sellers don't observe the quality), in which the seller's decision to continue to trade is a signal that enhances the seller's reputation. In our model with an outside option, entry decisions may carry information on seller's type because some low-type sellers may opt for the outside option rather than enter the platform.[5] In such cases, existence of an outside option instigates partial screening, lending new-comers to the platform the benefit of a higher initial reputation compared with the case without such an option. By contrast, if restarts are allowed in the same platform, genuine new starters inevitably suffer from a lower initial reputation inflicted by restarters who "contaminate" the pool of new-comers.

To highlight the core forces underlying the reputation mechanism delineated above, it is useful to reiterate the two ways in which a feedback system can facilitate information transmission. One is pure learning on the seller's true ability from observation of realized qualities. The other is endogenous intertemporal complementarity between a seller's honesty in pre-trade communication and the evolution of the market belief regarding his ability to deliver quality. This complementarity induces high-ability sellers to adopt a more trustworthy communication strategy, because they anticipate less bad news to disclose, and therefore find such a strategy to be less costly. It is via this second effect that feedback systems foster trust and credibility in pre-trade communication.

Observe that this second effect is possible because, in updating the market belief on sellers' ability, feedback comments are used relative to the truthfulness of sellers' announcements. Thus, our analysis suggests that there may be benefits in designing feedback sys-

[4] Potential ways to make restarting with a clean record more difficult and costly have been discussed by some authors, such as Friedman and Resnick (2001), who term the issue as "cheap pseudonyms," and Dellarocas (2006). It has not been shown hitherto, however, how damaging cheap pseudonyms may be to the functioning of feedback systems.

[5] Similar effects are present in Atkeson, Hellwig and Ordonez (2010) in a moral hazard context.

tems in a way that informs buyers about the verity of pre-trade announcements, as well as realized quality in past transactions.

Note that pre-trade communication would be useless without the feedback system due to the inherent conflict of interests between sellers and buyers. This proclaims a key characteristic of our model, namely, that pre-trade announcements cannot carry information on the item's quality unless they also carry information on the seller's ability. This implies that no pre-trade communication may arise if the seller's ability is known. It is also worth noting that adverse selection on an arbitrarily small difference in ability is enough to actuate credible communication via the reputation mechanism for sufficiently patient sellers.

To focus on the endogenous complementarity between trust and reputation that drives our mechanism, we adopt an adverse selection model in seller's ability. A side effect of this modeling choice is that communication adds no social value so long as all items are traded, precluding formal discussions of social welfare. However, we stress that our mechanism extends straightforwardly to a moral hazard setting where sellers differ in their cost of exerting effort that enhances the average quality of their items. As explained in the conclusion, our results in this setting indicate that pre-trade communication can motivate the more efficient sellers to exert high effort and thereby, enhance the social welfare. This finding complements Mailath and Samuelson (2001) who show, inter alia, that for reputation effects to arise in a similar model but without communication, the seller's type needs to be subject to continual random changes.

At a theoretical level, the current paper builds on the reputation literature initiated by Kreps and Wilson (1982) and Milgrom and Roberts (1982), and further developed by Diamond (1989), Fudenberg and Levine (1989), Mailath and Samuelson (2001), Ely and Valimaki (2003), and Cripps, Mailath and Samuelson (2004), among others.[6] In this literature, one type of agent is committed to a certain behavior and another, strategic type of agent either mimics this behavior so as to be pooled with the committed type (pooling motive), or diverts from it so as not to be mistaken for the committed type (separating motive). We do not assume a committed type in our model, yet both of the motives arise endogenously, the pooling (separating) motive for sellers of low (high) ability type.

Our paper contributes specifically to the literature on cheap-talk reputation which, due to the nature of the issue, often concerns experts, advisors and certifiers. Sobel (1985) shows that an "enemy" (an informed agent who has a completely opposing preference to the decision maker) may build a reputation by mimicking the honest reporting of a "friend" (who has a perfectly aligned preference with the decision maker), while Morris (2001) shows that even a friend may have a reputational incentive to lie. Ottaviani and Sorensen (2001, 2006) study reputational cheap talk in a different model where experts are motivated by exogenous career concerns.[7] Benabou and Laroque (1992) study the reputation of financial experts, while Mathis, McAndrews and Rochet (2009) examine the

[6]Mailath and Samuelson (2006) provide an extensive review of this literature.

[7]Although the mechanism differs for experts, they also find that often adverse selection on experts' quality enhances meaningful communication.

extent to which reputation concerns discipline rating agencies.[8]

Our model is akin to Sobel (1985) and Morris (2001) in that the model does not assume an inherently honest type. In contrast to these papers, however, a "friendly" seller cannot exist in our model because sellers have the same monotonic preferences over prices. As a consequence, knowing the seller's type would prevent any communication by even the high-type seller in our context, whilst it would trivially solve the problem for the friendly type in theirs. In this respect, our contribution differs substantially from theirs. Finally, the current paper also makes a methodological innovation in establishing the existence and uniqueness of equilibrium when Blackwell's condition for a contraction mapping does not apply.

The next section describes the baseline model and defines equilibrium. Section 3 presents some preliminary results. Section 4 analyzes the reputation mechanism and characterizes the unique honest equilibrium outcome. Section 5 analyzes an extended model in which sellers may freely opt out or restart with a new identity. Section 6 discusses some further extensions and concludes. The Appendix contains technical details.

2 Model

We consider a marketplace (or website) on which sellers interact with a large set of buyers in infinite periods $t = 1, 2, \cdots$. Initially, we focus on a single cohort of sellers all entering the marketplace at date $t = 1$ and staying there forever. This baseline model is analyzed until Section 4. Then, in Section 5 we introduce options for sellers to opt out for a fixed exit value or to freely restart with a clean record.

Each seller is either of a high-ability type ($\theta = h$) or a low-ability type ($\theta = \ell$) where $0 < \ell < h < 1$. Each seller's type is private information and fixed across periods. His perceived ability in each period t is captured by his *reputation* $\mu_t \in [0, 1]$, the common belief that prospective buyers attach to him being of a high type ($\theta = h$) in that period.

In each period t, a seller with reputation μ_t draws one item for sale of a random quality q_t which is good (g) with probability θ and bad (b) with probability $1 - \theta$ where $\theta \in \{h, \ell\}$ is his type. Observing the quality of the item, the seller publicly makes a cheap talk announcement $m_t \in \{G, B\}$ about its quality, where $m_t = G$ (B) is interpreted as announcing the quality to be g (b). We say that the agent *tells the truth* if he announces G when $q_t = g$ or B when $q_t = b$, and *lies* otherwise.[9]

Modelling a seller as above is in line with, for instance, online markets for collectibles and used goods, such as in eBay or Amazon, where the same seller repeatedly sells similar items of varying quality, often referred to as the "state" of the item; Websites such as Amazon.com and PriceMinister.com specifically require the sellers to choose within a set

[8] These papers use the adverse selection approach to reputation. Papers (on cheap-talk reputation) also exist that use the so-called "bootstrap" approach based on the folk theorem argument, e.g., Park (2005) on cheap talk reputation of differentiated experts and McLennan and Park (2007) on auditor reputation.

[9] Alternatively, we may model that each seller posts a price p at which buyers either buy or not, and the purchaser of the item reports whether satisfied ($q \geq p$) or not ($q < p$). Our equilibrium continues to be an equilibrium in this alternative model.

of pre-codified levels (new, like new, good, fair, etc.) to describe the state of their items.[10]

The quality of an item captures buyers' willingness to pay, which we normalize as $g = 1$ and $b = 0$. The prospective buyers are myopic and try to maximize the expected quality minus the price paid. We assume a competitive demand side so that each item is traded at a price that is equal to the expected quality.[11] In particular, sellers are not competing either because their goods are non-rival or because there are more buyers than sellers. (For instance, sellers run an auction amongst multiple buyers with the same valuation for the good and common beliefs.) At the end of each period, the purchaser observes the true quality q_t and honestly reports it publicly.[12] The seller's reputation is revised from μ_t to μ_{t+1} based on m_t and q_t, and the period $t+1$ starts. The seller's objective is to maximize the discounted sum of its revenue stream with discount factor $\delta \in (0, 1)$. In each period t, the full history of messages and items' quality delivered by the seller is publicly known. The structure of this game, denoted by Γ, is common knowledge.

Our equilibrium concept is Markov perfect equilibrium with state variable μ_t. Thus, seller's equilibrium strategy specifies the probability that a seller "lies" as a function of his type θ, the reputation level μ, and the quality q of the item drawn.

Given a seller's strategy, a "price profile" $p^*_m(\mu)$ is defined as the posterior probability that an item is of good quality ($q = 1$) when announced $m \in \{G, B\}$ by a seller with reputation μ. Being the expected quality, $p^*_m(\mu)$ is also the price at which the item will be traded.

A "transition rule" is a function $\pi^*_{mq}(\mu)$ that specifies the posterior probability that a seller is of a high type ($\theta = h$) in the next period, when in the current period his reputation is μ and he sells an item of quality q after announcing m.

In (Markov perfect) equilibrium, the seller's strategy is optimal given the price profile and the transition rule, where the price profile and the transition rule are obtained by Bayes rule from the seller's strategy whenever possible.

Before turning to the characterization of the equilibria with adverse selection and cheap-talk communication, we discuss a few properties of our model.

3 Preliminary considerations

The term "reputation" in the economic literature encompasses several notions, two of which are present in our model. Firstly, reputation may refer to the beliefs concerning the average quality provided by the seller to the market. In our model this corresponds to the beliefs μ_t on the seller's type θ. Secondly, reputation may refer to the level of confidence that consumers have on the truthfulness of the description of the good by a seller prior to sale. This notion thus refers more to trust than to beliefs on the type. As is shown below, however, these two concepts are intrinsically linked.

[10]Reputation effects in eBay have been empirically analyzed by, among others, Jin and Kato (2006) and Cabral and Hortacsu (2008).

[11]This is in line with Mailath and Samuelson (2001) and Bar-Isaac (2003).

[12]We assumed that the quality, although observable by the buyer, cannot be verified ex-post, so that no warranty contract is feasible.

We use the term "pure learning" to refer to the fact that mere observation of the history of quality q_t helps consumers improve their knowledge on the seller's type, in a non-strategic manner.

3.1 The pure learning outcome

Suppose that there is no communication, say because the seller doesn't observe the quality of the good. Then, in every period a seller's item is traded for a price that is equal to its expected quality

$$p_t = E(q|\mu_t) = \mu_t h + (1 - \mu_t)\ell$$

given his reputation μ_t, which evolves according to a simple Bayes rule based on the feedback report. The price and reputation follow a martingale, so they increase or decline depending on whether the quality delivered last period was good or bad.

This pure learning outcome is the equilibrium outcome of the so-called "babbling equilibrium" of the game Γ described in Section 2, which obtains for instance when the seller always announces G and thus, the message m_t, containing no informational content, is ignored. The price and reputation evolve as described above and since announcement doesn't affect the continuation game, it is trivially optimal for the seller to announce G.

3.2 The case of a single type

Next, suppose that there is a single possible type θ.[13] Due to risk-neutrality and certainty to trade, our model has the feature that in this case transmitting information to the buyer brings no future benefit to the seller. The reason is that since the price in period t will be the expected quality conditional on the information available then, the ex-ante expected price is θ in all future periods, and consequently, the seller's continuation payoff from the next period must be $\theta/(1-\delta)$ regardless of what he does now.

This implies that repeated interaction cannot elicit any information transmission through communication. To see this, suppose to the contrary that the message is informative in some period, which means that the probability of announcing G when $q = b$ is different from the probability of doing so when $q = g$ in that period. Thus, the prices would differ for the two messages. But, since the continuation payoff from the next period is $\theta/(1-\delta)$ independently of the message to be sent as verified above, the seller would announce with certainty the message that would fetch the highest price, irrespective of q, contradicting the supposition above that he would announce differently contingent on q.

Thus, when the type of the seller is common knowledge, there is no scope of meaningful communication in equilibrium.

[13]This differs from a situation that the buyers' beliefs assign probability 1 to θ because, as we shall see below, such a belief can be revised in off-equilibrium contingencies.

3.3 On communication in equilibrium

We say that an equilibrium involves communication if there is a positive probability that in some period the message conveys some information. In our model, there are two types of information that can be transmitted: information about the quality of the current item, q, and information about the seller's type $\theta \in \{h, \ell\}$.

In our set-up, the seller's message conveys information about q in some period with prevailing reputation level μ_t, if the two messages, G and B, generate different posterior beliefs on q by Bayes rule. This is the case if both messages are sent with positive probability and $p^*_G(\mu_t) \neq p^*_B(\mu_t)$ in equilibrium. Similarly, messages convey information about θ if the Bayesian updating of the seller's reputation will depend on the message to be sent by the seller as well as on the quality of the item to be reported.

We asserted above that communication about the quality of the item is not possible if there is a single type. This observation extends to the following property when there are multiple types, i.e., in a setting of adverse selection:

Property 1: Messages cannot convey information on the quality of the good unless they convey information on the type of the seller.

To see this, suppose that no information on θ is transmitted by messages in equilibrium. Then, a seller's reputation will be updated as a function only of the history of delivered quality as explained above and consequently, in any period t the expected future payoff of the seller is independent of the current message to be sent. If $p^*_G(\mu_t) \neq p^*_B(\mu_t)$, therefore, both types of seller must send the same message (the one that fetches a higher price) with probability 1 regardless of q, establishing that the messages be uninformative on q.

Therefore, adverse selection and signalling about the type are necessary ingredients for messages to be a credible signal of quality in our environment.

4 Honest Equilibrium

We now turn to the analysis of the equilibrium with communication. Given seller's strategy and the associated price profile p^*_m and transition rule π^*_{mq}, we define the value function $V^*_\theta(\mu) : [0, 1] \to \mathbb{R}$, as the expected discounted sum of the revenue stream of a seller of type θ and reputation μ. As we wish to study the extent to which reputation motives help to induce truthful revelation of the quality of the product, we focus on equilibria with the following property:

Condition H: An h-type seller always tells the truth regardless of q for all $\mu > 0$.

We refer to an equilibrium satisfying the condition H as an "honest equilibrium," although we will also use "equilibrium" without qualification when there is no ambiguity. As some results are rather technical, we derive them formally in Appendix and present them below in a more heuristic manner.

First, observe that it is impossible for an ℓ-type seller to be always truthful in equilibrium, because then the price reflects quality perfectly and each seller gets exactly for what he delivers, so an ℓ-type seller has nothing to lose by cheating. To see this note that, since the unconditional expected price at any date is at least ℓ (because the expected quality in the market is equal to some weighted average of h and ℓ), the continuation value is at least $\ell/(1-\delta)$ for any level of reputation μ. But this lower bound, $\ell/(1-\delta)$, would be the equilibrium payoff of an ℓ-type seller if he were to always tell the truth, because then in each period he would get an equilibrium price $p_G^* = 1$ with probability ℓ and $p_B^* = 0$ with probability $1-\ell$. Thus, an ℓ-type seller would lie and announce G upon drawing a bad quality item.

However, there should be no incentive to misreport good quality as bad, since this would reduce the current price without enhancing next period's reputation. Indeed, an ℓ-type seller is truthful whenever $q = g$ as we verify in Appendix (Lemma 3). Thus, given Condition H, we characterize the equilibrium strategy by the probability, denoted $y^*(\mu)$, that an ℓ-type seller lies when $q = b$ and his reputation is μ.[14] Moreover, $y^*(\mu) > 0$ for all μ in any honest equilibrium as we outline below (and formally prove in Appendix). Thus, the price profile p_m^* and the transition rule π_{mq}^* are determined as explained below.

For $\mu \in [0,1]$, the expected quality of an item claimed as good ($m = G$) by a seller with reputation μ is

$$p_G(\mu, y) := \frac{\mu h + (1-\mu)\ell}{\mu h + (1-\mu)(\ell + (1-\ell)y)}, \tag{1}$$

if an ℓ-type seller would falsely claim so with a probability $y \in [0,1]$. So, equilibrium prices are

$$p_G^*(\mu) = p_G(\mu, y^*(\mu)) \quad \text{and} \quad p_B^*(\mu) = 0. \tag{2}$$

For $y > 0$ and $\mu < 1$, $p_G(\mu, y)$ is strictly increasing in μ and strictly decreasing in y. Thus, the more an ℓ-type seller lies, the lower is the short-term gain from lying.

Next, we explain the transition rule $\pi_{mq}^*(\mu)$. Let us define

$$\pi_{Bb}(\mu, y) := \frac{\mu(1-h)}{\mu(1-h) + (1-\mu)(1-\ell)(1-y)}. \tag{3}$$

Then, in cases of truth-telling, Bayesian updating of reputation prescribes

$$\pi_{Gg}^*(\mu) = \frac{\mu h}{\mu h + (1-\mu)\ell} \quad \text{and} \quad \pi_{Bb}^*(\mu) = \pi_{Bb}(\mu, y^*(\mu)) \tag{4}$$

whenever they are well-defined. Since π_{Bb} is strictly increasing in $\mu < 1$ and $y < 1$, the more an ℓ-type seller lies, the higher is the gain in reputation from telling the truth.

In cases of falsely claiming good quality, we have $\pi_{Gb}^*(\mu) = 0$ by Bayes rule as long as $\mu < 1$. Note that $\pi_{Gb}^*(1)$ is undefined by Bayes rule, but it needs to be sufficiently low for an h-type seller with $\mu = 1$ to not lie upon drawing $q = b$ as per the condition H. Without loss of generality, we take the convention that $\pi_{Gb}^*(1) = 0$ because it ensures continuity of equilibrium variables V_θ^* and π_{Gb}^* at $\mu = 1$, without affecting the characterization of equilibrium outcome (Cf. Lemma 4 in Appendix). This means that in any honest

[14] In light of Condition H, we will also specify later the probability that an h-type seller lies when $\mu = 0$ and $q = b$, but this is off-equilibrium and thus, is inconsequential.

equilibrium, a substantial, if not as drastic, drop in reputation needs to be postulated in the off-equilibrium contingency that a seller with reputation $\mu = 1$ lies. One interpretation is that buyers, having classified the seller as h-type with certainty based on past records, would reconsider their interpretation of the records upon (hypothetical) arrival of new evidence inconsistent with this classification.

Lastly, $\pi^*_{Bg}(\mu)$ is undefined since a good quality item is never claimed to be bad, but setting $\pi^*_{Bg}(\mu) = 0$ does not affect incentive compatibility. Summarizing, we set

$$\pi^*_{Gb}(\mu) = \pi^*_{Bg}(\mu) = 0 \quad \text{for all} \quad \mu \in [0, 1]. \tag{5}$$

4.1 Announcement strategy and value function of ℓ-type

We start with an ℓ-type seller with extreme reputation levels. Once a seller's reputation falls to $\mu = 0$, he cannot increase his reputation above 0, because Bayes rule dictates that $\pi^*_{mq}(0) = 0$ for any message $m \in \{G, B\}$ that is sent with a positive probability. Therefore, a seller with reputation 0 announces the message that gives the highest price regardless of q, which implies that the seller gets the same equilibrium price, ℓ, regardless of q. This is the case when an ℓ-type seller's announcement strategy is independent of q when $\mu = 0$. Since labeling of messages is inconsequential due to the costless nature of cheap talk messages, we make the convention that an ℓ-type seller announces G regardless of q when $\mu = 0$, i.e., $y^*(0) = 1$. This confirms that $p^*_G(0) = \ell \geq p^*_B(0)$. An immediate consequence is that the equilibrium value at $\mu = 0$ is $V^*_\ell(0) = \ell/(1-\delta)$.

Consider an ℓ-type seller with the other extreme reputation $\mu = 1$. In this case, he should lie with probability one upon drawing $q = b$, i.e., $y^*(1) = 1$, because otherwise his value would be $V^*_\ell(1) = \ell + \delta V^*_\ell(1)$ since $p^*_G(1) = 1$ and thus, $V^*_\ell(1) = \ell/(1-\delta) = V^*_\ell(0)$, contradicting the hypothesis that he lies with probability less than one. Note that this conclusion relies on $\ell/(1-\delta)$ being the lower bound of continuation payoff as asserted earlier and thus, is valid regardless of the reputation level that lying would take to, $\pi^*_{Gb}(1)$. Given $\pi^*_{Gb}(1) = 0$ as postulated earlier without loss of generality, we have $V^*_\ell(1) = 1 + \delta\big(\ell V^*_\ell(1) + (1-\ell)V^*_\ell(0)\big)$ so that

$$V^*_\ell(1) = V^*_\ell(0) + \Delta \quad \text{where} \quad \Delta := \frac{1-\ell}{1-\delta\ell} < 1. \tag{6}$$

As asserted above, an honest equilibrium is characterized by a function $y^*(\mu)$, specifying the probability of an ℓ-type seller lying when $q = b$, that is optimal relative to the associated price profile p^*_m, transition rule π^*_{mq}, and the value function V^*_ℓ where, given (2),

$$\begin{aligned} V^*_\ell(\mu) \quad &= \quad [\ell + (1-\ell)y^*(\mu)]p_G(\mu, y^*(\mu)) \\ &\quad +\delta\big(\ell V^*_\ell(\pi^*_{Gg}(\mu)) + (1-\ell)[y^*(\mu)V^*_\ell(0) + (1-y^*(\mu))V^*_\ell(\pi_{Bb}(\mu, y^*(\mu)))]\big). \end{aligned} \tag{7}$$

This means that

$$y^*(\mu) \in \arg\max_{0\leq y\leq 1} yp_G(\mu, y^*(\mu)) + \delta y V^*_\ell(0) + (1-y)\,\delta V^*_\ell(\pi_{Bb}(\mu, y^*(\mu))). \tag{8}$$

Note that V_ℓ^* determines y^* by (8), which in turn determines V_ℓ^* by (7). Thus, V_ℓ^* is a fixed point of a mapping defined by (7) and (8) as explained below, which we denoted by T. The following lemma is useful in this discussion.

Lemma 1 *In any honest equilibrium, V_ℓ^* is continuous and strictly increasing in $\mu \in [0,1]$.*

Proof. See Appendix (within Proof of Proposition 1). ■

In light of Lemma 1, we could define T for all continuous and increasing functions $V : [0,1] \to \mathbb{R}$ such that $V(0) = \ell/(1-\delta)$ and $V(1) = \ell/(1-\delta) + \Delta$. However, this is not very useful because the operator T turns out to fail the Blackwell's sufficiency condition for a contraction, which is a standard way of obtaining a unique solution in the literature. Thus, we define T for a domain $\mathcal{F}$ of all non-decreasing and right-continuous functions V on $[0,1]$ with $V(0) = \ell/(1-\delta)$ and $V(1) = \ell/(1-\delta) + \Delta$, which is compact (under weak topology) so that a suitable Fixed-Point Theorem may be applied.

We now explain how to define the mapping T. Observe that $p_G(\mu, y)$ is continuous and strictly decreasing in y (except when $\mu = 1$) and $V(\pi_{Bb}(\mu, y))$ increases in y. For any continuous function $V \in \mathcal{F}$, therefore, there is a unique function $y_V : [0,1] \to [0,1]$, referred to as the "pseudo-best-response," defined by

$$y_V(\mu) = \begin{cases} 0 & \text{if} \quad p_G(\mu, 0) < \delta(V(\pi_{Bb}(\mu, 0)) - V(0)) \\ 1 & \text{if} \quad p_G(\mu, 1) > \delta(V(\pi_{Bb}(\mu, 1)) - V(0)) = \delta\Delta \\ y & \text{s.t.} \quad p_G(\mu, y) = \delta(V(\pi_{Bb}(\mu, y)) - V(0)), \text{ otherwise.} \end{cases} \tag{9}$$

We extend the pseudo-best-response to discontinuous functions $V \in \mathcal{F}$ in Appendix.

The term on the RHS of the (in)equalities in (9) is the gain from enhanced reputation that accrues to a seller who truthfully announces bad quality, which is bounded by $\delta\Delta$. Hence, $y_V(\mu)$ is optimal for ℓ-type seller relative to V, $p_G(\mu, y_V(\mu))$ and $\pi_{Bb}(\mu, y_V(\mu))$. Since $p_G(\mu, 0) = 1 > \delta\Delta$, it must be the case that $y_V(\mu) > 0$ for all μ (as asserted earlier). From $p_G(1,1) = 1 > \delta\Delta$, it further follows that $y_V(\mu) = 1$ for all $\mu > \bar{\mu}$ where

$$\bar{\mu} := \inf \{\mu \in [0,1] \mid p_G(\mu, 1) > \delta\Delta\} \;<\; 1. \tag{10}$$

Note that the threshold $\bar{\mu}$ may be zero, in which case an ℓ-type seller lies whenever $q = b$.

Since $y_V(\mu) > 0$ for all μ, i.e., it is optimal for an ℓ-type seller to lie whenever $q = b$, we now define a mapping $T : \mathcal{F} \to \mathcal{F}$ by

$$T(V)(\mu) := p_G(\mu, y_V(\mu)) + \delta \left(\ell V(\pi_{Gg}^*(\mu)) + (1-\ell)V(0) \right). \tag{11}$$

The equilibrium value function V_ℓ^* is a fixed point of the operator T and the equilibrium strategy is $y^*(\mu) = y_{V_\ell^*}(\mu)$, which is characterized below.

Proposition 1 *There exists a unique fixed point V_ℓ^* of T. The value function V_ℓ^* is continuous and strictly increasing on $[0,1]$; and the strategy $y^* = y_{V_\ell^*}$ is continuous on $[0,1]$.*

For smooth flow of discussion, we relegate a detailed proof to Appendix. Basically, the existence follows from continuity of the operator T and the Fan-Glicksberg Fixed Point Theorem. Uniqueness is obtained separately from existence by using the properties of the fixed point. It stems from the observation that $V_\ell^*(\mu)$ is uniquely determined for $\mu > \bar{\mu}$, from which the values for lower μ are uniquely traced back recursively via (11).

Thus, our result is distinguished from that of Benabou-Laroque (1992) who obtain existence and uniqueness by applying Blackwell's Theorem.[15] In our model T is not non-decreasing in V and hence, Blackwell's condition for a contraction is not applicable. Mathis, McAndrew and Rochet (2009) exploit an idea akin to ours in obtaining a constructive proof of existence in a model of rating agencies. Their proof relies on the fact that only positive claims generate trade and thus can be verified, which simplifies the analysis greatly.

4.2 Optimality for h-type sellers and equilibrium

To verify equilibrium conditions fully, it remains to show that it is indeed optimal for an h-type seller to announce truthfully, given the strategy y^*, price profile p_m^*, and the transition rule π_{mq}^* identified in the previous section. Recall V_h^* denotes the value function of an h-type seller.

Note that $\pi_{Bb}^*(0)$ is undefined by Bayes rule. We proceed presuming that $\pi_{Bb}^*(0) = 0$, which ensures incentive compatibility at $\mu = 0$ and thus, incurs no loss of generality in characterizing the equilibrium outcomes (as elaborated in Appendix). Then, it is clear that an h-type seller would lie upon drawing $q = b$ if his reputation is $\mu = 0$ (which is an off-equilibrium contingency).[16] Thus,

$$V_h^*(0) = \frac{\ell}{1-\delta} \quad \text{and} \quad V_h^*(1) = \frac{h}{1-\delta}. \tag{12}$$

An h-type seller, if he followed the strategy of an ℓ-type seller, would obtain a higher expected payoff because he would get a better sequence of draws on average, i.e., $V_h^*(\mu) > V_\ell^*(\mu)$ for $\mu > 0$. This means that the value of maintaining reputation is higher for h-type seller than for ℓ-type seller. But, the value from burning it by falsely claiming good quality is the same for both types at $p_G^*(\mu)+\delta V_h^*(0)$. Hence, whenever an ℓ-type seller is indifferent between lying and not, which is the case when $\mu < \bar{\mu}$ and $q = b$, an h-type seller prefers to announce truthfully. By the same token, whenever an ℓ-type seller prefers to tell the truth, which is the case when $q = g$, so does an h-type seller.

It remains to consider the case that an h-type seller draws $q = b$ when $\mu \geq \bar{\mu}$. In this case, he gets $\delta V_h^*(1)$ by announcing truthfully and $p_G^*(\mu) + \delta V_h^*(0)$ by announcing untruthfully because $\pi_{Bb}^*(\mu) = 1$ and $\pi_{Gb}^*(\mu) = 0$. Thus, it is optimal for an h-type seller

[15] Morris (2001) and Bar-Isaac (2003) also use Blackwell's Theorem.

[16] This is an artefact of assuming $\pi_{Bb}^*(0) = 0$, and is not essential. It is possible that $\pi_{Bb}^*(0) > 0$, hence $V_h^*(0) > V_\ell^*(0)$ in equilibrium, which would imply that h-seller is presumed to announce $q = b$ truthfully when $\mu = 0$ (see Proof of Theorem 1 in Appendix). Since what h-seller would do when $\mu = 0$ is postulation of off-equilibrium behavior anyway, such an equilibrium does not affect the set of equilibrium outcomes.

to announce truthfully if and only if $\delta(V_h^*(1)-V_h^*(0)) \geq \max_{\mu\geq\bar{\mu}} p_G^*(\mu) = 1$, or equivalently,

$$h-\ell \geq \frac{1-\delta}{\delta} \quad \Longleftrightarrow \quad \delta \geq \delta_h := \frac{1}{h-\ell+1}. \tag{13}$$

The analysis up to now is summarized below as the first main result.

Theorem 1 *There exists an equilibrium satisfying Condition H if and only if $\delta \geq \delta_h$. The equilibrium outcome is unique.*

Proof. In Appendix. ∎

4.3 Properties of the equilibrium

The honest equilibrium exhibits several interesting features which we discuss below (and prove in Appendix). We also highlight the contrast with the pure learning outcome (babbling equilibrium) in price and learning dynamics.

First, the value of building reputation is bounded for ℓ-type sellers however patient they may be (whereas it tends to infinity for h-type sellers as δ approaches 1):

Property 2 The value of good reputation for an ℓ-type seller, Δ, is smaller than the good/bad quality differential, $g-b=1$.

To understand this property, recall that an ℓ-type seller strictly prefers to lie when $\mu=1$ in the honest equilibrium. This is because otherwise, the value at $\mu=1$ would be equal to that at $\mu=0$, removing any incentive to be truthful. This means that the short-term gain from lying when $\mu=1$, which is $g-b$, exceeds the value of lost reputation, $\delta\Delta$. The willingness to pay for high reputation of an ℓ-type seller known as such, Δ, is decomposed as follows: moving from $\mu=0$ to $\mu=1$ raises the expected price of the first period to g from $(1-\ell)b+\ell g$, and it yields $\delta\Delta$ in future value when the current quality turns out to be g, which occurs with probability ℓ. Thus, $\Delta = g-(1-\ell)b-\ell g+\ell\delta\Delta < g-b = 1$.

Note that the higher is the probability with which ℓ-type seller lies, the lower is the short-term gain from cheating because the current price is lower, while the reputational gain from staying truthful is higher.

We established that if the reputation is high enough, in particular, if $\mu \geq \bar{\mu}$, then an ℓ-type seller lies with certainty upon drawing a bad quality item because even the smallest short-term gain, $p_G(\mu,1)$, exceeds the maximum possible reputational gain, $\delta\Delta$. Note from (10) that $\bar{\mu}>0$ if and only if $\delta\Delta > \ell = p_G(0,1)$, or equivalently,

$$\delta \;>\; \delta_\ell := \frac{\ell}{1-\ell+\ell^2}. \tag{14}$$

The same does not hold for $\mu \in (0,\bar{\mu})$. So, the probability of lying should be reduced until the short-term gain grows enough to balance out the potential reputational gain. This reduction of lying probability cannot go on all the way to 0 because $\delta\Delta$, the maximum possible reputational gain, is bounded away from the short-term gain obtainable when ℓ-type seller were fully honest by Property 2. Consequently, there is a uniform upper bound on how honest ℓ-type sellers may be regardless of δ, as is verified in Appendix:

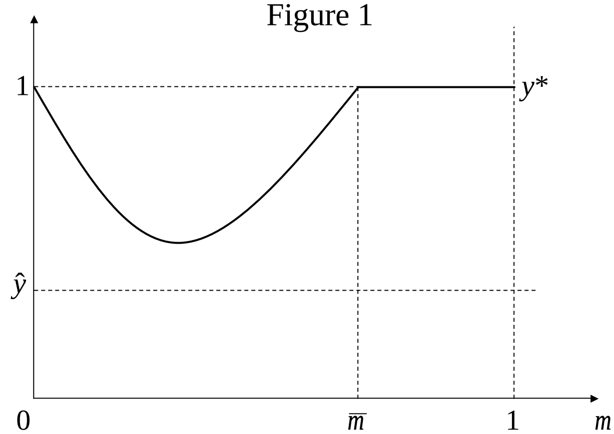
$$y^*(\mu) > \hat{y} := \frac{h-\ell}{1-\ell} \quad \text{for all } \mu. \tag{15}$$

Thus, the market never trusts sellers at a level approaching full confidence however patient they may be. In conjunction with (14), therefore, we obtain

Property 3 If $\delta \le \delta_\ell$, then $y^*(\mu) = 1$ for all $\mu \in [0,1]$. If $\delta > \delta_\ell$, then $y^*(\mu)$ is a continuous function assuming values $y^*(\mu) = 1$ at $\mu = 0$ and $\mu \ge \bar{\mu}$ and $y^*(\mu) \in (\hat{y}, 1)$ at all $\mu \in (0, \bar{\mu})$.

Figure 1 illustrates typical $y^*(\mu)$ for $\delta > \delta_\ell$.

Figure 1



The result (15) places an upper bound on the market price for items claimed to be good: $p_G^*(\mu) < p_G(\mu, \hat{y}) = \mu + (1-\mu)\frac{\ell}{h}$. However, the aforementioned insight that price and reputation are correlated is confirmed by the following finding (proved in Appendix):

Property 4 $p_G^*(\mu)$ is strictly increasing in μ for $\mu < 1$.

This property, that higher reputation results in higher prices, is at the heart of our equilibrium analysis. It is the main driver of the incentives to build reputation via honest communication with buyers. Moreover, as intuition suggests, the more patient sellers are the more trustworthy they become at all levels of reputation, as stated in the next property (proved in Appendix). Consequently, more patient sellers fetch higher price by announcing good quality.

Property 5 $\bar{\mu}$ increases in δ; $p_G^*(\mu)$ (resp. $y^*(\mu)$) strictly increases (resp. decreases) in δ for all $\mu \in (0, \bar{\mu})$.

Notice that our model exhibits a trade-off between short-term transmission of information on q via honest announcements of sellers, and the speed of learning on θ through the observation of past records: When $y^*(\mu)$ decreases, the price $p^*_G(\mu)$ increases but the updated reputation level after an honest announcement of a bad quality, $\pi^*_{Bb}(\mu)$, is lower. Indeed, truthful announcements of bad quality carry less information on the seller's type since false announcements are less frequent. Thus, increasing the discount factor fosters credible communication at a cost of slower learning.

In any case, the lower bound on $y^*(\mu)$ in (15) implies a lower bound on the informational content on the seller's type conveyed by equilibrium messages. In particular, trustworthy behavior of sellers is always "good news" for their reputation:

Property 6 $\pi^*_{Bb}(\mu) > \mu$ all $\mu > 0$.

If $\delta_h \leq \delta \leq \delta_\ell$, therefore, the equilibrium path has a simple characterization: a seller lies if and only if he is ℓ-type and draws a bad quality item. Hence, for any seller the trading price p_t of period t increases over time until he draws a bad item, at which point his type is revealed. Then, p_t drops to ℓ for good if $\theta = \ell$; whilst $p_t = q_t$ in all subsequent periods if $\theta = h$, i.e., p_t reflects the true quality.

If $\delta > \max\{\delta_h, \delta_\ell\}$, on the other hand, an ℓ-type seller with a reputation below $\bar{\mu}$ randomizes between announcing truthfully and untruthfully upon drawing $q = b$. As long as he tells the truth, he builds reputation and thus benefits from higher future prices for items he will announce to be good.

In either case, therefore, the seller's reputation increases over time until one of two events occurs: *i*) the seller falsely announces $m = G$ when $q = b$, in which case his type is revealed to be ℓ and the price drops to ℓ for good; or *ii*) the seller truthfully announces $m = B$ when his reputation is $\mu > \bar{\mu}$, in which case his type is revealed to be h and the price reflects the true quality in the future. In particular, the seller's type gets revealed whenever a bad quality item is drawn once his reputation exceeded $\bar{\mu}$. Since the reputation level goes above μ within a finite number of periods unless it collapses to zero due to false announcement, we have:

Property 7 The seller's type is revealed within finite time with probability 1.

As such, a salient characteristic of the equilibrium with credible communication is that information on seller's type is revealed much more quickly than in the case without communication, where convergence occurs only asymptotically. Compared to the case of pure learning in Section 3.1, a key difference concerns updating of reputation following a bad draw ($q = b$). In the pure learning outcome it is given by $\mu_{t+1}(b) = \pi_{Bb}(\mu_t, 0) < \mu_t$. In the honest equilibrium, reputation improves as long as the seller announces truthfully. While bad quality is always interpreted as "bad news" in the absence of communication, it is perceived as "good news" in our model if truthfully announced and facilitates learning. Since this extra learning effect stems from h-type seller's desire to separate through more trustworthy behavior, communication helps to mitigate the asymmetric information problem along two interrelated dimensions: *i*) it helps credible communication of the true quality, and *ii*) it helps consumers learn the true type of the seller. As explained in the Introduction, these extra effects are possible because the feedback comments are utilized to infer the accuracy of pre-trade communication as well as the delivered quality. This insight should be useful for designing effective feedback systems.

The price dynamics also differ substantially. In an honest equilibrium, the price for items announced to be good increases over time roughly in line with the reputation until the seller's type is revealed, while the price stays constant at zero for items announced as bad. In contrast, the price in the pure learning outcome follows a martingale where the price of date t is independent of the realization of the quality at date t.

We have fully characterized the honest equilibrium above. What kind of other equilibria may there be? We already discussed the babbling equilibrium. Since the pre-trade announcement is cheap talk, it is not surprising that other equilibria exist. For instance, babbling may prevail for some reputation levels whilst the honest equilibrium prevails for other reputation levels. In particular, for any $\mu' \in (0, 1)$ the following is easily verified to be an equilibrium from the analysis in this section: the announcements are completely ignored when $\mu \leq \mu'$ and the seller adopts the strategy in the honest equilibrium for $\mu > \mu'$.

One cannot preclude the possibility that still other kinds of equilibria may exist, such as those in which an h-type seller is not always honest and/or more than two messages are used. If two distinct messages, say m and n, are used by an h-type seller who is not fully honest at some reputation μ, then he sends both messages with positive probabilities for at least one quality, say $q = g$. This indifference implies that $p^*_m(\mu) + \delta V^*_h(\pi^*_{mg}(\mu)) = p^*_n(\mu) + \delta V^*_h(\pi^*_{ng}(\mu))$ where $\pi^*_{mg}(\mu) \neq \pi^*_{ng}(\mu)$, say $\pi^*_{mg}(\mu) < \pi^*_{ng}(\mu)$. If $\pi^*_{ng}(\mu) < 1$, then an ℓ-type seller also mixes between sending m and n when $q = g$, further implying that $V^*_h(\pi^*_{ng}(\mu)) - V^*_h(\pi^*_{mg}(\mu)) = V^*_\ell(\pi^*_{ng}(\mu)) - V^*_\ell(\pi^*_{mg}(\mu))$. These are additional restrictions on equilibrium value functions that are absent for the honest equilibrium. Whether these additional restrictions may be satisfied for a non-trivial set of μ's in an equilibrium, is a complex question that is beyond the scope this paper.

5 Outside Option and New-life

Up to now we have assumed that sellers stay in one marketplace forever and that memory is infinite. One of the key issues surrounding the reputation mechanisms based on feedback systems is that sellers may find ways to escape from the bad consequences of damaged reputation. For instance, sellers may change to another marketplace. As emphasized for instance by Friedman and Resnick (2001), even within a given marketplace it may be difficult to keep track of the identity of a seller, in which case a seller may have at any date an option to erase his history by changing his identity and start again as a new seller.[17]

One may then be concerned that such a possibility may destroy the fundamental reputation mechanism of feedback systems elaborated in Section 4. We show in this section

[17] The ability to do so depends on the technology used by the platforms. This is known to be an issue with eBay for instance (Delarocas 2006), but would be less of an issue when the platform controls bank coordinates or social status of companies, for then it would involve creating a new firm which is costly.

that this is not the case, although the effectiveness of the mechanism in fostering honest communication by the seller is reduced.

Addressing this issue requires delineating additional equilibrium dynamic interactions because the incentive to change identity depends on the market's belief concerning new sellers, and this belief depends on equilibrium strategies.

Thus, we extend the analysis to a full dynamic setting by augmenting our baseline model with a stationary entry and exit of sellers in every period: There is a constant measure 1 of sellers on the platform in each period. Each seller dies with probability $\chi \in (0, 1)$ at the end of each period. These deaths are replaced by measure χ of newborn sellers at the beginning of the next period. Each new born seller is of h-type with probability $\mu^i \in (0, 1)$. We maintain the assumptions that there is a single platform to which each seller brings an item (of either good or bad quality) for sale in each of infinite periods and that the past record of each seller is publicly known.

Additionally, at the beginning of each period, each seller has an option of escaping from further consequences of his reputation. We distinguish two ways in which this may happen. In the first case, which we refer to as the model "with an outside option," sellers have an option to exit the platform for another activity that yields an exogenously given value v_o. Thus, a seller joins and remains in the platform only if his equilibrium expected payoff from doing so is larger than v_o.

In the second case, which we refer to as the "new-life" model, sellers cannot exit the platform but may acquire a new identity in any period and restart as if a newborn seller. So, a "new seller" may be either a newborn or a restarter and buyers cannot distinguish them. In stationary equilibria of this model, a unique initial value for all new sellers is endogenously determined and any seller would restart as a new seller if the equilibrium value associated with his current reputation level falls below the initial value. Thus, from the perspective of each seller, restarting is equivalent to exiting the platform for an outside option of this initial value.

5.1 Equilibrium with an outside option

Suppose that sellers, instead of being able to restart, may take an outside option of a fixed value $v_o > 0$ and leave the market at the beginning of any period. We assume that this decision is irreversible so that a seller who exits never reenters (this would be the case for instance if there are many platforms that do not allow reentry). Moreover, we assume that the exit value is the same for both types of sellers, although our results extend straightforwardly to type-dependent exit values provided that the difference is not too large (see our 2009 working paper).

Note that the analysis is equivalent to that in Section 3 if $v_o \leq \ell/(1-\delta)$ because then sellers never exit the platform. Indeed, the equilibrium derived there applies to each seller, starting from the date of birth when his initial reputation starts at μ^i and period t is interpreted as the age or seniority of the seller (the number of trading periods since joining the platform). Since all sellers would exit immediately if $v_o \geq 1/(1-\delta)$ as will be verified below, we focus on $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$ in the sequel.[18]

Given that age is observed, we can analyze this case by examining the equilibrium for a representative seller who is born in period 1. An equilibrium then prescribes an entry decision at birth, and announcement strategy and exit decision of the seller conditional on entry. As before, focusing on stationary equilibria with state variable μ_t, we are concerned with the existence and properties of equilibrium that satisfies Condition H.

As will be outlined below, the analysis and main results are similar to those presented in Section 3. But, we need additionally to take care of the seller's entry and exit decisions. Since equilibrium value functions are increasing in the reputation level, a seller would exit if his reputation falls below the threshold level at which the equilibrium value is lower than v_o (or would not enter if the initial reputation is below this threshold level). In order to determine this threshold level endogenously, we need to compare the option of exiting with that of not exiting in the current period for sellers of all possible reputation levels.

To this end, we first characterize equilibrium of an instrumental "auxiliary model" in which a newborn seller must trade in the platform in the first period but may exit in any future period for an outside option value v_o, and his initial reputation level is randomly assigned to be any level between 0 and 1. Since the initial reputation level may take any value, we define and characterize the value function and the strategy for all levels of $\mu \in [0, 1]$ in this auxiliary game.

Then, by incorporating equilibrium entry decisions into this analysis, we establish the existence of a stationary equilibrium of the model with an outside option, i.e., when every seller decides whether to enter the platform or not upon birth with an initial reputation level μ^i (Theorem 2), and compare with the case without an outside option (Proposition 2).

First, we show that the auxiliary model described above has an equilibrium:

Lemma 2 *In the auxiliary model described above with $v_o \in \big(\frac{\ell}{1-\delta}, \frac{1}{1-\delta}\big)$, a stationary equilibrium exists that satisfies Condition H if $\delta\big(\frac{h}{1-\delta} - v_o\big) > 1$.*

We defer a formal proof to Appendix since it is analogous to that of Theorem 1 with v_o replacing $V_\ell^*(0)$. In particular, the same argument as before verifies straightforwardly that an ℓ-type seller truthfully announces when $q = g$. Thus, as before the equilibrium strategy is described by the equilibrium probability of lying by an ℓ-type seller upon drawing $q = b$, denoted by $y^\dagger(\cdot)$ to distinguish it from that in Section 3. Let $V_\theta^\dagger$ denote the value function of the equilibrium identified in Lemma 2. Then, the boundary values are routinely computed to be

$$V_\ell^\dagger(0) \;=\; \ell + \delta v_o \in \Big(\frac{\ell}{1-\delta}, v_o\Big) \quad \text{and} \quad V_\ell^\dagger(1) = v_o + \Delta_{v_o} \;>\; V_\ell^*(1) \tag{16}$$

$$\text{where} \quad \Delta_{v_o} \;:=\; \frac{1-(1-\delta)v_o}{(1-\delta\ell)} \;<\; \Delta\,.$$

Note also that $V_\ell^\dagger(1) < v_o$ if $v_o \geq 1/(1-\delta)$ and thus, all sellers would exit immediately as asserted above.

[18]Note that the upper bound, $1/(1-\delta)$, exceeds $V_\ell^*(1)$. This is because as v_o increases, so does ℓ-seller's value at $\mu = 1$, $V_\ell^\dagger(1)$ defined below, maintaining $v_o < V_\ell^\dagger(1)$ as long as $v_o < 1/(1-\delta)$.

By (16), there is a threshold level $\mu_o^\dagger$, defined by $V_\ell^\dagger(\mu_o^\dagger) = v_o$. Furthermore, analogously to Lemma 1, $V_\ell^\dagger$ can be shown to be continuous and strictly increasing. Thus, in any non-initial period an ℓ-type seller would choose to exit the market if and only if his prevailing reputation level is below $\mu_o^\dagger$. As before, every truthful announcement increases the reputation until there is a lie by an ℓ-type seller. It is also verified in the proof of Lemma 2 that from any reputation level, a truthful announcement of bad quality pushes up the reputation level above $\mu_o^\dagger$ in the next period, i.e., for $t > 1$, $\mu_t > \mu_o^\dagger$ or $\mu_t = 0$. In the auxiliary model, therefore, whatever the initial reputation level is at birth, a newborn seller continues to trade in the platform until he lies on the quality for the first time, after which he exits because his type is revealed to be ℓ.

We now return to the model with an outside option, postulating that once on the platform, the seller follows the strategy $y^\dagger(\cdot)$ derived in the auxiliary game. Then each seller, who is of h-type with probability μ^i, has to choose at birth whether to enter the platform or to opt for an outside value of v_o and leave the market for good. It is straightforward that if $\mu^i > \mu_o^\dagger$, there exists an equilibrium in which every seller enters irrespective of his type because then the initial reputation level is μ^i and consequently, it is optimal for both types of seller to enter because $V_h^\dagger(\mu^i) > V_\ell^\dagger(\mu^i) > V_\ell^\dagger(\mu_o^\dagger) = v_o$. But, if $\mu^i < \mu_o^\dagger$ then both types entering with certainty is not viable since then it would be suboptimal for an ℓ-seller to enter because $V_\ell^\dagger(\mu^i) < V_\ell^\dagger(\mu_o^\dagger) = v_o$. In this case, a seller randomizes between entering and not if he is ℓ-type, in such a way that the initial reputation is $\mu_o^\dagger$, as elaborated below.

Theorem 2 *In the model with an outside option where $v_o \in \left(\frac{\ell}{1-\delta}, \frac{1}{1-\delta}\right)$, there exists a stationary equilibrium satisfying Condition H if $\delta\left(\frac{h}{1-\delta} - v_o\right) > 1$, in which an h-type seller enters with probability 1 at birth and the initial reputation upon entry is $\mu_1 = \max\{\mu^i, \mu_o^\dagger\}$.*

Proof. We already established above that both types entering for sure is an equilibrium if $\mu^i > \mu_o^\dagger$. The same holds if $\mu^i = \mu_o^\dagger$. For $\mu^i < \mu_o^\dagger$, consider the entry strategy that a seller enters for sure if h-type but enters with probability $\frac{\mu^i(1-\mu_o^\dagger)}{\mu_o^\dagger(1-\mu^i)} \in (0,1)$ if ℓ-type. Then the initial reputation upon entry is $\mu_o^\dagger$ by Bayes rule. Conditional on entry, it constitutes a continuation equilibrium for both types to behave according to the equilibrium identified in Lemma 2. Moreover, it is optimal for an ℓ-type seller to randomize between entering and not at birth because $V_\ell^\dagger(\mu_o^\dagger) = v_o$, while it is optimal for an h-type seller to enter for sure because $V_h^\dagger(\mu_o^\dagger) > v_o$. This completes the proof. ■

Thus, a new feature of the equilibrium with an outside option is that, when the proportion of ℓ-type is high among newborn sellers, the initial entry stage induces some screening through self-selection.[19] As a consequence, the expected payoff at birth is $\max\{V_\ell^\dagger(\mu^i), v_o\}$ for an ℓ-type seller and $\max\{V_h^\dagger(\mu^i), V_h^\dagger(\mu_o^\dagger)\}$ for an h-type seller.

While the outside option affects the values of equilibrium variables, the nature of equilibrium upon entry remains unchanged from the case of no outside option. In particular, every truthful announcement increases a seller's reputation until there is a lie followed by an exit. But, the level of trust placed on announcements is lower as explained below.

[19] An implication is that a platform may be able to improve screening by charging a positive price to join.

The equations in (16) indicate that the extreme values are higher for ℓ-type seller when there is an outside option: $V_\ell^\dagger(\mu) > V_\ell^*(\mu)$ for $\mu = 0, 1$. However, the value of a good reputation (the difference of the two extreme values) is lower and as a consequence, an ℓ-type seller lies more. In particular, it remains to be the case that $y^\dagger(\mu) = 1$ above some critical level of reputation, but this threshold is lower than its counterpart without an outside option, $\bar{\mu}$. Furthermore, below this level an ℓ-type seller lies more frequently on quality, as formally stated below.

Proposition 2 *For* $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$,

$$y^\dagger(\mu) > y^*(\mu)\ \forall \mu \in (0, \bar{\mu}) \quad \text{and} \quad y^\dagger(\mu) = y^*(\mu) = 1\ \forall \mu \in [\bar{\mu}, 1]. \tag{17}$$

Proof. In Appendix. ■

Thus, availability of an option to exit and obtain an outside value (that is larger than the value attached to bad reputation), results in a uniform increase in the probability that a bad item is falsely claimed as good. As a consequence, the price for items announced as good is lower at all levels of reputation. At the same time, note that learning takes place faster than in the equilibrium without such an option for two reasons. First, it is more likely that an ℓ-type seller reveals his type by falsely announcing good quality. Second, reputation gets updated to higher levels following truthful announcements of bad quality, which also implies that a reputation level is reached sooner at which the seller's type is revealed for sure if a bad item is drawn (because an ℓ-type seller would definitely lie).

As pointed out earlier, our model exhibits a balance between the short-run reliability of communication and the speed of revelation of the seller's type over time. The outside option shifts this balance toward faster separation of types. The impact on h-type sellers' welfare is ambiguous, however, since they benefit from fast reputation building but suffer from lower prices for given levels of reputation.

5.2 New-life

We now extend the analysis to the new-life model, in which there is no outside option but sellers can freely restart in any period by erasing their history and obtaining a new identity. We assume that buyers cannot distinguish a newborn seller from an old seller restarting with a clean record, and focus on stationary equilibria satisfying Condition H, that is, the proportion of ℓ-type sellers restarting afresh is constant every period, while h-type sellers never lie and consequently, never change their identities.

In a stationary equilibrium, a constant mass, denoted by χ_1, of new sellers (newborns and restarters) appear on the platform in each period and they start with a reputation level set at an endogenous default level, denoted by μ_1, which reflects the mix of genuine newborn sellers (of which a proportion μ^i are of type h) and equilibrium mass of sellers who restart. Let v_1 denote the value of an ℓ-type seller starting at the "default reputation level" μ_1. Notice that the value function depends on v_1, while the fraction μ_1 depends on both the value function and v_1.

To determine μ_1 and v_1 endogenously, we start by treating v_1 as a parameter representing an outside option value to be obtained when an ℓ-type seller exits the market, so that we can apply the result from the previous section to determine the equilibrium strategy and value function of ℓ-type seller that are consistent with v_1, which we denote by $y^\dagger_{v_1}$ and $V^\dagger_{v_1}$, respectively, to emphasize their dependence on v_1.

From Section 5.1 we know that in the equilibrium identified in Lemma 2 (with v_1 playing the role of v_o), an ℓ-type seller will change identity only when his reputation has fallen to $\mu_t = 0$ after a lie. Based on this, we derive in Appendix the proportion $\Lambda(v_1)$ of ℓ-type sellers starting at a given period, who will change identity and restart in some future period as

$$\Lambda(v_1) := (1-\ell)(1-\chi)\sum_{k=1}^{\infty}\sum_{\mathbf{h}^k\in H^k}\Pr(\mathbf{h}^k)y^\dagger_{v_1}(\pi(\mathbf{h}^k)). \tag{18}$$

Here, $\Pr(\mathbf{h}^k)$ is the ex ante probability that an ℓ-type seller remains in the platform without having cheated after a k-period quality history $\mathbf{h}^k \in H^k := \{g, b\}^k$ according to $y^\dagger_{v_1}$; and $\pi(\mathbf{h}^k)$ is posterior reputation for a seller who has survived the history $\mathbf{h}^k$ without cheating, updated according to $y^\dagger_{v_1}$ from initial reputation μ_1.

Since the mass of new ℓ-type sellers at each date is $\chi_1(1-\mu_1)$, the mass of sellers who change their identities in a stationary equilibrium is $\chi_1(1-\mu_1)\Lambda(v_1)$. Therefore, stationarity dictates that the mass of new sellers is

$$\chi_1 = \chi + \chi_1(1-\mu_1)\Lambda(v_1). \tag{19}$$

Since only ℓ-type sellers restart, Bayes rule dictates that in a stationary state

$$\mu_1 = \frac{\chi\mu^i}{\chi_1}. \tag{20}$$

Solving (19) and (20) simultaneously, we define a mapping $\mu_1^\dagger : (\frac{\ell}{1-\delta}, \frac{1}{1-\delta}) \to (0,1)$ as

$$\mu_1^\dagger(v_1) \; := \; \frac{\mu^i - \mu^i\Lambda(v_1)}{1-\mu^i\Lambda(v_1)} \; < \; \mu^i \tag{21}$$

where the inequality follows from $0 < \Lambda(v_1) < 1$. This mapping determines the unique initial reputation level of new sellers, that is consistent with a given value v_1 of a new seller of type ℓ. Thus, the equation $v_1 = V^\dagger_{v_1}(\mu_1^\dagger(v_1))$ must hold in a stationary equilibrium. By solving this equation for v_1, we show that

Theorem 3 *If sellers can freely change identity, there exists a stationary equilibrium satisfying Condition H if h and δ are sufficiently large (but less than 1). In this equilibrium, the default reputation level is lower than μ^i and ℓ-type sellers lie more than they do when identities cannot be changed.*

Proof. In Appendix. ■

The main conclusion is thus that allowing sellers to change identity at no cost doesn't invalidate our result that equilibria exist in which sellers of high ability always communicate truthfully. Of course it impedes somewhat the power of incentives: Since Proposition 2 applies in any stationary equilibrium, sellers' announcements are less reliable than when fresh restart with a new identity is not possible. However, this does not mean that untruthful announcements are more frequent in the market when restarts are possible than when they are not: ℓ-type sellers who have lied once, rather than keep lying forever when $q = b$, would start afresh and announce according to $y^{\dagger}_{v_1}(\mu_1)$. In fact, when δ is close to 1 there will be more truthful announcements in the market when sellers are allowed to restart with a new identity.

Nevertheless, h-type sellers tend to suffer more due to untrustworthy behavior of ℓ-type sellers when restarts are possible for two reasons. First, as in the case of an outside option, the reduced reliability of ℓ-type sellers results in lower prices for h-type sellers. Notice that the price profile is the same as that when the seller had an exogenous outside option of value $v_o = v_1$. Second, newborn h-type sellers suffer from depressed reputation at birth due to the restarters boosting the fraction of ℓ-type in the pool of new sellers ($\mu_1 < \mu^i$).

In particular, contrary to the case of an outside option, the decline in confidence and in prices doesn't necessarily get translated to a faster revelation of the seller's type. Since sellers start with a lower initial reputation level, it may take longer for an h-type seller to be identified as such by the market. Overall, the possibility to restart one's activity on a platform with a new identity reduces the credibility of communication without necessarily enhancing the separation dynamics of seller types.

6 Concluding Remarks

In this paper we investigate the extent to which the quality of product can be credibly communicated to prospective buyers in experience good markets. We show that if there is adverse selection on seller's ability (in supplying good quality items), credible communication can be sustained by reputational motives in spite of the inherent conflict of interests between sellers and buyers. In addition, if sellers can restart with a new identity, a stationary equilibrium exists but the reliability of sellers' announcements deteriorates uniformly across all reputation levels.

To focus on the reputational incentives in pre-trade communication, we carried out our analysis in a model of pure adverse selection on seller's ability. However, the analysis can be extended to situations that involve moral hazard. To see this, modify the baseline model in such a way that in each period a seller draws an item of good quality with a probability h if he exerted high effort at a cost of $c_\theta > 0$ that depends on the seller's type $\theta \in \{h, \ell\}$, but he draws a good item with a probability ℓ if he exerted low effort at zero cost. Note that our honest equilibrium continues to be an equilibrium in this modified model if c_h is small enough for an h-type seller to find it worthwhile to exert high effort, but c_ℓ is large so that an ℓ-type seller finds otherwise.[20] If pre-trade communication is not

[20] This is the case if $\delta(V^*_h(\pi_{Bb}(\mu, y^*(\mu)) - \frac{c_h}{1-\delta} - V^*_h(0)) \geq p_G(\mu, y^*(\mu))$ for all μ, and the inequality

possible, this model is equivalent to the baseline model of Mailath and Samuelson (2001) without replacement of types, for which they show that high effort cannot be induced unless discontinuous strategies are allowed (Proposition 2, p424). Our result suggests that pre-trade communication may motivate the more efficient type to exert high effort by facilitating the learning process in the market.

We anticipate that our analysis can be extended in other directions as well. For instance, in the context of internet markets, to examine the effect of competition between trading websites appears as an interesting task from the market design perspective, particularly because rival websites would influence each other by providing exit values as our results suggest. Analysis of such competition may also carry implications on the market segmentation between trading websites and their pricing strategies. At the same time, it also seems fruitful to explore other routes that might enhance the value of online reputation, for instance, via creating a market for trading online identities *a la* Tadelis (1999).

APPENDIX

A. Proof of Proposition 1

For completeness, we state and prove (as Lemmas) all the assertions made without full verifications in the informal analysis preceding the statement of Proposition 1. Recall that in the first paragraph of Section 4.1 we proved that $V_\ell^*(0) = \ell/(1-\delta)$ and adopted the convention that $y^*(0) = 1$ without loss of generality. Note that this conclusion is valid in all equilibria of Γ, i.e., without imposing any restrictions (not even the condition H).

Since, for any μ, some weighted average of $p_G^*(\mu)$ and $p_B^*(\mu)$ is equal to the expected quality $h\mu + \ell(1-\mu)$, any seller can get a price no lower than ℓ in every period, and a price strictly greater than ℓ in the first period if $\mu > 0$. Therefore,

$$V_\ell^*(\mu) \;>\; V_\ell^*(0) \;=\; \frac{\ell}{1-\delta} \qquad \forall \mu > 0. \tag{22}$$

Let $z^*(\mu)$ denote the equilibrium probability that an ℓ-seller of reputation μ lies upon drawing $q = g$.

Lemma 3 *In any honest equilibrium, $z^*(\mu) = 0$ for all $\mu \in [0,1]$ and $V_\ell^*(1) \geq \overline{V}_\ell := \frac{\ell}{1-\delta} + \Delta = \frac{1-\delta(1-\ell+\ell^2)}{(1-\delta)(1-\delta\ell)} > V_\ell^*(\mu)$ for all $\mu < 1$.*

Proof. We already established that $y^*(0) = 1$ and $z^*(0) = 0$ by convention in the first paragraph of Section 4.1. First, we show that $p_G^*(\mu) > p_B^*(\mu)$ for all $\mu \in (0,1]$. This is immediate if $y^*(\mu) = 0$ or $\mu = 1$, because then $p_G^*(\mu) = 1$ and $p_B^*(\mu) < 1$ by the condition

is reversed if c_ℓ replaces c_h, where $V_h^*(0) = \frac{\ell}{1-\delta} = V_\ell^*(0)$ and y^*, V_ℓ^* and V_h^* are as derived in Section 4. Such values of c_h and c_ℓ exist because *i)* $\delta(V_h^*(\pi_{Bb}(\mu, y^*(\mu)) - V_h^*(0)) > p_G(\mu, y^*(\mu))$ for $\mu \geq \bar{\mu}$ if $\delta > \delta_h$ due to (13), *ii)* $\delta(V_\ell^*(\pi_{Bb}(\mu, y^*(\mu)) - V_\ell^*(0)) = p_G(\mu, y^*(\mu))$ by definition of V_ℓ^*, and *iii)* $V_h^*(\pi_{Bb}(\mu, y^*(\mu)) > V_\ell^*(\pi_{Bb}(\mu, y^*(\mu)) + \zeta$ for some $\zeta > 0$ due to (47), continuity of V_θ^* for $\mu > 0$, and $\lim_{\mu\to 0} \pi_{Bb}(\mu, y^*(\mu)) > 0$, where the last inequality is implied by $\lim_{\mu\to 0} \delta\big(V_\ell^*(\pi_{Bb}(\mu, y^*(\mu)) - V_\ell^*(0)\big) = \ell$.

H and Bayes rule. If $y^*(\mu) > 0$ for $\mu \in (0,1)$, lying is no worse than telling the truth when $q = b$ so that $p^*_G(\mu) + \delta V^*_\ell(\pi^*_{Gb}(\mu)) \geq p^*_B(\mu) + \delta V^*_\ell(\pi^*_{Bb}(\mu))$, while $\pi^*_{Gb}(\mu) = 0 < \pi^*_{Bb}(\mu)$. These two inequalities, together with (22), imply that $p^*_G(\mu) > p^*_B(\mu)$ as desired.

Now, to prove the Lemma by contradiction, suppose $z^*(\mu) > 0$ for some $\mu > 0$. If $\mu < 1$, this would imply that $\pi^*_{Bg}(\mu) = 0 < \mu < \pi^*_{Gg}(\mu)$ by condition H and Bayes rule. Then, due to (22) and $p^*_G(\mu) > p^*_B(\mu)$ shown above, we would have $p^*_G(\mu) + \delta V^*_\ell(\pi^*_{Gg}(\mu)) > p^*_B(\mu) + \delta V^*_\ell(\pi^*_{Bg}(\mu))$, i.e., telling the truth would be strictly better than lying when $q = g$. Since this would contradict $z^*(\mu) > 0$, we conclude that $z^*(\mu) = 0$ for all $\mu \in (0,1)$.

At this point, note that an ℓ-seller with reputation $\mu = 1$ can guarantee a value of $\overline{V}_\ell = \frac{1+\delta(1-\ell)V^*_\ell(0)}{1-\delta\ell}$ by lying if and only if $q = b$. Thus, $V^*_\ell(1) \geq \overline{V}_\ell > V^*_\ell(0)$.

We now show that $\overline{V}_\ell > V^*_\ell(\mu)$ $\forall \mu < 1$. To reach a contradiction, suppose $\overline{V}_\ell \leq V^*_\ell(\mu)$ for some $\mu < 1$. First, consider the case that $\sup_{\mu<1} V^*_\ell(\mu) > \overline{V}_\ell$. Then, one can find arbitrary small $\epsilon > 0$ and $\mu_\epsilon < 1$ such that

$$V^*_\ell(\mu_\epsilon) > \sup_{\mu<1} V^*_\ell(\mu) - \epsilon > \overline{V}_\ell + \frac{\delta}{1-\delta\ell}\epsilon > V^*_\ell(0) + \frac{\delta}{1-\delta}\epsilon. \tag{23}$$

Since $z^*(\mu_\epsilon) = 0$ as shown above, if $y^*(\mu_\epsilon) = 0$ then $V^*_\ell(\mu_\epsilon) = \ell p^*_G(\mu_\epsilon) + \delta(\ell V^*_\ell(\pi^*_{Gg}(\mu_\epsilon)) + (1-\ell)V^*_\ell(\pi^*_{Bb}(\mu_\epsilon))) \leq \ell + \delta \sup_{\mu<1} V^*_\ell(\mu) < \ell + \delta(V^*_\ell(\mu_\epsilon) + \epsilon) \;\Rightarrow\; V^*_\ell(\mu_\epsilon) < V^*_\ell(0) + \frac{\delta}{1-\delta}\epsilon$, contradicting (23); If $y^*(\mu_\epsilon) > 0$, on the other hand, $V^*_\ell(\mu_\epsilon) = p^*_G(\mu_\epsilon) + \delta(\ell V^*_\ell(\pi^*_{Gg}(\mu_\epsilon)) + (1-\ell)V^*_\ell(0)) < 1 + \delta(\ell(V^*_\ell(\mu_\epsilon) + \epsilon) + (1-\ell)V^*(0)) \;\Rightarrow\; V^*_\ell(\mu_\epsilon) < \overline{V}_\ell + \frac{\delta}{1-\delta\ell}\epsilon$, violating (23).

It remains to consider the case that $\sup_{\mu<1} V^*_\ell(\mu) = \overline{V}_\ell$, so that $\overline{V}_\ell = V^*_\ell(\tilde{\mu})$ for some $\tilde{\mu} < 1$. Since $z^*(\tilde{\mu}) = 0$ as shown above, if $y^*(\tilde{\mu}) = 0$ then $V^*_\ell(\tilde{\mu}) = \ell p^*_G(\tilde{\mu}) + \delta(\ell V^*_\ell(\pi^*_{Gg}(\tilde{\mu})) + (1-\ell)V^*_\ell(\pi^*_{Bb}(\tilde{\mu}))) \leq \ell + \delta V^*_\ell(\tilde{\mu}) \;\Rightarrow\; V^*_\ell(\tilde{\mu}) \leq V^*_\ell(0)$, contradicting (22); If $y^*(\tilde{\mu}) > 0$, on the other hand, $V^*_\ell(\tilde{\mu}) = p^*_G(\tilde{\mu}) + \delta(\ell V^*_\ell(\pi^*_{Gg}(\tilde{\mu})) + (1-\ell)V^*_\ell(0)) < 1 + \delta(\ell V^*_\ell(\tilde{\mu}) + (1-\ell)V^*(0)) \;\Rightarrow\; V^*_\ell(\tilde{\mu}) < \overline{V}_\ell$, contradicting $V^*_\ell(\tilde{\mu}) = \overline{V}_\ell$ asserted above.

Thus, we have proved $\overline{V}_\ell > V^*_\ell(\mu)$ for all $\mu < 1$. Finally, together with $V^*_\ell(1) \geq \overline{V}_\ell$ shown above, this dictates that $z^*(1) = 0$ because $p^*_G(1) + \delta V^*_\ell(\pi^*_{Gg}(1)) = 1 + \delta V^*_\ell(1) > p^*_B(1) + \delta V^*_\ell(\pi^*_{Bg}(1))$. ∎

For $\mu \in (0,1)$, because of (22) and Lemma 3, $y^*(\mu) = 0$ would imply that the short-term gain from lying when $q = b$, which is $p^*_G(\mu) - p^*_B(\mu) = 1$, would exceed the long-term loss, $\delta\big(V^*_\ell(\pi^*_{Bb}(\mu)) - V^*(\pi^*_{Gb}(\mu))\big) < \delta\Delta$ because $\Delta < 1$ as asserted in (6), contradicting the optimality of $y^*(\mu) = 0$. Hence, given the convention $y^*(0) = 1$ adopted earlier, we have

$$y^*(\mu) > 0 \quad \forall \mu \in [0,1) \tag{24}$$

Furthermore, in light of Lemma 3 and Bayes rule, without loss of generality we set

$$p^*_G(0) = \ell \quad \text{and} \quad p^*_B(\mu) = \pi^*_{Gb}(\mu) = \pi^*_{Bg}(\mu) = 0 \;\; \forall \mu \in [0,1] \tag{25}$$

as asserted in the main text, subject to the following caveat: Determining $\pi^*_{Gb}(1)$ is a little delicate because it determines the value of $V^*_\ell(1)$ and thereby, the optimality of $y^*(\mu) = 1$ for $\mu < 1$ via determining the deviation value when an ℓ-seller with reputation μ announced truthfully upon drawing $q = b$, which would induce reputation $\pi^*_{Bb}(\mu) = 1$. The value of

$\pi^*_{Gb}(1)$ also plays a salient role for a seller of "maximal reputation" $\mu = 1$, who, upon drawing $q = b$, has a choice between maintaining its reputation with a low current price ($p^*_B(1) = 0$), and a high current price of $p^*_G(1) = 1$ followed by a drop of future payoff from $V^*_\theta(1)$ to $V^*_\theta(\pi^*_{Gb}(1))$ due to lost reputation.

However, notice that $\mu = 1$ for $\theta = \ell$ and $\pi^*_{Gb}(1)$ occur only off the equilibrium path, which allows us to conclude:

Lemma 4 *For any honest equilibrium, there exists an honest equilibrium with the same strategy of seller (i.e., the same $y^*(\mu)$) and $\pi^*_{Gb}(1) = 0$, so that, in particular, $V^*_\ell(1) = \overline{V}_\ell$.*

Proof. First, we prove $y^*(1) = 1$. To do so, suppose otherwise, i.e., $y^*(1) < 1$. Then, telling the truth would be optimal when $\mu = 1$ and $q = b$, so that $1 + \delta V^*_\ell(\pi^*_{Gb}(1)) \leq \delta V^*_\ell(\pi^*_{Bb}(1)) = \delta V^*_\ell(1)$. Thus,

$$\begin{aligned} & V^*_\ell(1) = \ell\big(1 + \delta V^*_\ell(1)\big) + (1-\ell)\big(y^*(1)(1 + \delta V^*_\ell(\pi^*_{Gb}(1)) + (1 - y^*(1))\delta V^*_\ell(1)\big) \\ \Rightarrow\ \ & V^*_\ell(1) \leq \ell\big(1 + \delta V^*_\ell(1)\big) + (1-\ell)\delta V^*_\ell(1) = \ell + \delta V^*_\ell(1) \ \Rightarrow\ V^*_\ell(1) \leq \ell/(1-\delta), \end{aligned}$$

contradicting (22). This proves $y^*(1) = 1$.

Now, given any honest equilibrium, suppose we reset $\pi^*_{Gb}(1) = 0$ without changing $y^*(\cdot)$. Then, the value $V^*_\ell(1)$ is lower and equal to $\overline{V}_\ell$, while $V^*_\ell(\mu)$ is unchanged for $\mu < 1$ (because $V^*_\ell(1)$ does not affect $V^*_\ell(\mu)$ for $\mu < 1$). Hence, the incentive compatibility for an ℓ-seller is preserved for $\mu < 1$. It is preserved for $\mu = 1$ as well since $1 + \delta V^*_\ell(0) > \delta\overline{V}_\ell$.

For an h-seller, the value function $V^*_h(\cdot)$ is unchanged. Since $\pi^*_{Gb}(1)$ is relevant only for the off equilibrium contingency that an h-seller lies when $\mu = 1$, it only remains to verify that the incentive compatibility condition continues to hold at $\mu = 1$. For this, note (i) from (24) that the incentive compatibility for $\mu < 1$ is $\delta V^*_h(\pi^*_{Bb}(\mu)) \geq p^*_G(\mu) + \delta V^*_h(0)$, and ($ii$) $p^*_G(\mu) \to 1$ and $\pi^*_{Bb}(\mu) \to 1$ as $\mu \to 1$ and consequently, the expected discounted sum of future prices from truth-telling, $V^*_h(\mu)$, is left-continuous at $\mu = 1$. Hence, it follows that $\delta V^*_h(1) \geq p^*_G(1) + \delta V^*_h(0)$, confirming the incentive compatibility at $\mu = 1$. ∎

Consequently, we focus on equilibria with $\pi^*_{Gb}(1) = 0$ and $V^*_\ell(1) = \overline{V}_\ell$ in the sequel, without loss of generality in the sense that it is inconsequential for the equilibrium outcome.

Given (25), as discussed prior to (6), we have

$$V^*_\ell(0) = \frac{\ell}{1-\delta} \quad \text{and} \quad V^*_\ell(1) = V^*_\ell(0) + \Delta \quad \text{where} \quad \Delta = \frac{1-\delta}{1-\delta\ell} < 1. \tag{26}$$

Since $y^*(\mu) > 0\ \forall \mu < 1$ as per (24) and $y^*(1) = 1$ as shown above, any equilibrium value function V^*_ℓ should satisfy

$$V^*_\ell(\mu) = p_G(\mu, y^*(\mu)) + \delta\big(\ell V^*_\ell(\pi^*_{Gg}(\mu)) + (1-\ell)V^*_\ell(0)\big) \quad \forall \mu \in [0,1]. \tag{27}$$

Let $\pi^1_{Gg}(\mu) = \pi^*_{Gg}(\mu)$ and $\pi^t_{Gg}(\mu) = \pi^*_{Gg}\big(\pi^{t-1}_{Gg}\big)$ recursively for $t \geq 2$ so that

$$\pi^t_{Gg}(\mu) = \frac{\mu h^t}{\mu h^t + (1-\mu)\ell^t} > \mu. \tag{28}$$

Then, expanding (27) by applying an analogous equation to $V_\ell^*(\pi_{Gg}^*(\mu))$ repeatedly,

$$\begin{aligned} V_\ell^*(\mu) &= \left[\sum_{t=0}^{\infty} \ell^t \delta^t p_G\left(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\right)\right] + \delta V_\ell^*(0)(1-\ell)\sum_{t=0}^{\infty} \ell^t\delta^t && (29)\\ &= \sum_{t=0}^{\infty} \delta^t \ell^t \big(p_G(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))) - \ell\big) + V_\ell^*(0). && (30)\end{aligned}$$

The next lemma describes an ℓ-seller's equilibrium behavior for large μ.

Lemma 5 *For any honest equilibrium, there exists $\bar{\mu} < 1$ (defined in (10)) such that $y^*(\mu) = 1$ if $\mu > \bar{\mu}$. Furthermore, $V_\ell^*(\mu)$ is continuous and strictly increasing on $(\bar{\mu}, 1]$ (with $\pi_{Gb}^*(1) = 0$).*

Proof. For sufficiently large $\mu < 1$, the payoff from lying when $q = b$ is arbitrarily close to 1 while that from telling the truth is bounded above by $\delta(\overline{V}_\ell - V_\ell^*(0)) = \delta\Delta < 1$. This means that $y^*(\mu) = 1$ for sufficiently large μ.

Since $p_G(\mu, y)$ is decreasing in y, we deduce that $y^*(\mu) = 1$ so long as $p_G(\mu, 1) \geq \delta\Delta$, i.e., for all $\mu \geq \bar{\mu}$ where $\bar{\mu}$ is defined in (10). Then, since $y^*(\pi_{Gg}^t(\mu))) = 1$ for all t for any $\mu \geq \bar{\mu}$ by (28), (29) uniquely determines $V_\ell^*(\mu)$ for $\mu \geq \bar{\mu}$ as

$$V_\ell^*(\mu) = \left[\sum_{t=0}^{\infty} \ell^t \delta^t p_G\left(\pi_{Gg}^t(\mu), 1\right)\right] + \delta V_\ell^*(0)(1-\ell)\sum_{t=0}^{\infty} \ell^t\delta^t,$$

which is continuous and strictly increasing in $\mu > \bar{\mu}$ because both $p_G(\mu, 1)$ and $\pi_{Gg}^*(\mu)$ are continuous and strictly increasing in μ. Moreover, $\lim_{\mu\to 1} V_\ell^*(\mu) = \overline{V}_\ell$, verifying continuity at $\mu = 1$. ■

The next lemma is useful in proving that V_ℓ^* is continuous and increasing (Lemma 1).

Lemma 6 *In any honest equilibrium, if $y^*(\mu)$ is continuous on an interval $(\mu_0, 1]$ then $\pi_{Bb}(\mu, y^*(\mu)) > \mu$ for all $\mu \in (\mu_0, 1)$, and $p_G^*(\mu)$ and $V_\ell^*(\mu)$ are continuous and strictly increasing in $\mu \in (\mu_0, 1]$.*

Proof. The proof is immediate if $\mu_0 \geq \bar{\mu}$ by Lemma 5. Hence, we consider the case that $\mu_0 < \bar{\mu}$ below. Let $\mu' \in (\mu_0, \bar{\mu}]$ be such that $V_\ell^*(\mu)$ is continuous and strictly increasing in $\mu \in (\mu', 1]$. It exists by Lemma 5. We proceed by showing that the properties in Lemma 6 hold on $(\mu' - \varepsilon, 1]$ for small enough $\varepsilon > 0$, in three steps as below.

Step 1: First, we verify that

$$\pi_{Bb}(\mu, y^*(\mu)) > \mu \quad \forall \mu \in [\mu', 1). \tag{31}$$

To reach a contradiction, suppose to the contrary that there is some $\mu \in [\mu', 1)$ such that $\pi_{Bb}(\mu, y^*(\mu)) \leq \mu$. Then, by continuity of $\pi_{Bb}(\mu, y)$ and $y^*(\mu)$ on $(\mu_0, 1]$, we can define

$$\tilde{\mu} = \max\{\mu < 1 \mid \pi_{Bb}(\mu, y^*(\mu)) \leq \mu\} \;\in\; [\mu', \bar{\mu}). \tag{32}$$

Here, $\tilde{\mu} \in [\mu', \bar{\mu})$ follows because for $\mu \in [\bar{\mu}, 1)$, by Lemma 5, $y^*(\mu) = 1$ and thus, $\pi_{Bb}(\mu, y^*(\mu)) = 1 > \mu$. By continuity, we also have $\pi_{Bb}(\tilde{\mu}, y^*(\tilde{\mu})) = \tilde{\mu}$.

It is easily verified from (3) that

$$\pi_{Bb}(\mu, y) \geq \mu \iff y \geq \hat{y} = (h - \ell)/(1 - \ell), \tag{33}$$

which implies that $y^*(\tilde{\mu}) = \hat{y}$ and thus

$$p_G(\tilde{\mu}, \hat{y}) = \delta(V_\ell^*(\tilde{\mu}) - V_\ell^*(0)). \tag{34}$$

Note from (28) that $\pi_{Gg}^t(\tilde{\mu}) > \tilde{\mu}$ for $t \geq 1$ and thus, $\pi_{Bb}(\pi_{Gg}^t(\tilde{\mu}), y^*(\pi_{Gg}^t(\tilde{\mu}))) > \pi_{Gg}^t(\tilde{\mu})$ by (32). Consequently, $y^*(\pi_{Gg}^t(\tilde{\mu})) > \hat{y}$ by (33). Therefore, since $p_G(\mu, y) \leq 1$ and $p_G(\mu, y)$ decreases in y, (30) implies that

$$V_\ell^*(\tilde{\mu}) - V_\ell^*(0) < \sum_{t=0}^{\infty} \big(p_G(\pi_{Gg}^t(\tilde{\mu}), \hat{y}) - \ell\big)\delta^t \ell^t. \tag{35}$$

Since $p_G(\mu, \hat{y}) = \big(\mu(h - \ell) + \ell\big)/h$ from (1) and (33), we further deduce from (35) that

$$\begin{aligned} V_\ell^*(\tilde{\mu}) - V_\ell^*(0) &< p_G(\tilde{\mu}, \hat{y}) - \ell + \sum_{t=1}^{\infty} \Big(\frac{\pi_{Gg}^t(\tilde{\mu})(h - \ell) + \ell(1 - h)}{h}\Big)\delta^t \ell^t \qquad (36) \\ &< p_G(\tilde{\mu}, \hat{y}) - \ell + \sum_{t=1}^{\infty} \Big(\frac{(h - \ell) + \ell(1 - h)}{h}\Big)\delta^t \ell^t \\ &= p_G(\tilde{\mu}, \hat{y}) - \ell + (1 - \ell)\frac{\delta\ell}{1 - \delta\ell} \\ &= p_G(\tilde{\mu}, \hat{y}) - \frac{(1 - \delta)\ell}{1 - \delta\ell} < p_G(\tilde{\mu}, \hat{y}) \end{aligned}$$

where the second inequality follows from $\pi_{Gg}^t(\tilde{\mu}) < 1$. Thus, we have reached a contradictory conclusion that (34) cannot hold at $\tilde{\mu}$. This completes the proof of (31).

Step 2: We now show that

[A] $p_G^*(\mu)$ and $V_\ell^*(\mu)$ are continuous and strictly increasing in $\mu \in (\mu' - \varepsilon, 1]$ for sufficiently small $\varepsilon > 0$.

For $\mu > \mu' - \varepsilon$, let us define y_μ as the unique solution of $\pi_{Bb}(\mu, y_\mu) = \mu'$. For sufficiently small $\varepsilon > 0$, (31) and continuity of $y^*(\mu)$ imply that $y^*(\mu) > y_\mu$ for all $\mu > \mu' - \varepsilon$.

Consider the graph of $\delta(V_\ell^*(\pi_{Bb}(\mu, \cdot)) - V_\ell^*(0))$ as a function of y, called the "loss graph." Because $V_\ell^*(\mu)$ is continuous and strictly increases in $\mu \in (\mu', 1]$ and $\pi_{Bb}(\mu, y) > \mu'$ for all $\mu > \mu' - \varepsilon$ and $y > y_\mu$ by definition of y_μ, for $\mu > \mu' - \varepsilon$ and $y > y_\mu$ the loss graph:

i) is continuous and strictly increases in y with terminal value $\delta(V_\ell^*(1) - V_\ell^*(0)) = \delta\Delta$;

ii) continuously "shifts downward" as μ decreases.

On the other hand, the graph of $p_G(\mu, \cdot)$ as a function of $y \in [0, 1]$, which we call the "gain graph,"

iii) is continuous and strictly decreases in y for $y < 1$, and

iv) continuously shifts downward as μ decreases.

In addition, when $\mu = \bar{\mu}$ the two graphs touch at $y = y^*(\bar{\mu}) = 1$.

Consider $\mu \in (\mu' - \varepsilon, \bar{\mu})$. Given that $y^*(\mu) > y_\mu$ for $\mu \in (\mu' - \varepsilon, \bar{\mu}]$, which is implied by (31) and continuity of y^* at μ' for sufficiently small $\varepsilon > 0$, the loss graph and the gain graph cross at a unique point in $(y_\mu, 1]$, which must be equal to $y^*(\mu)$ because $y^*(\mu) > y_\mu$.

Since, in the domain $y > y_\mu$, the two graphs shift continuously as μ changes in $(\mu' - \varepsilon, \bar{\mu}]$, both this unique intersection point, $y^*(\mu)$, and the value of the gain graph at the intersection point, $p_G^*(\mu)$, change continuously in $(\mu' - \varepsilon, \bar{\mu}]$. Furthermore, since the two graphs strictly shift upward as μ increases in $(\mu' - \varepsilon, \bar{\mu}]$, it follows that $p_G^*(\mu)$ strictly increases in $\mu \in (\mu' - \varepsilon, \bar{\mu}]$. Consequently, in conjunction with $y^*(\mu) = 1$ for $\mu \geq \bar{\mu}$, we have verified that $p_G^*(\mu)$ is continuous and strictly increasing in $\mu \in (\mu' - \varepsilon, 1]$. Equation (30) ensures that the same property holds for $V_\ell^*(\mu)$ and thereby, [A] above.

Step 3: As the last step of the proof, let $\underline{\mu}' \geq \mu_0$ be the infimum of $\mu' \in (\mu_0, \bar{\mu}]$ such that $V_\ell^*(\mu)$ is continuous and strictly increasing in $\mu \in (\mu', 1]$. If $\underline{\mu}' > \mu_0$, (31) and [A] would lead to a contradiction to the fact that $\underline{\mu}'$ is the infimum of such points, completing the proof. ■

Proof of Lemma 1. First, we show that y^* is continuous at all $\mu \in (0, 1]$. We already showed that it is continuous on $[\mu_0, 1]$ for some $\mu_0 < \bar{\mu}$ in the preceding proof.

To reach a contradiction, suppose y^* is discontinuous at some points and let $\mu^d \in (0, \mu_0]$ be the supremum of these points. Then y^* is continuous at all $\mu > \mu^d$. By Lemma 6, $\pi_{Bb}(\mu, y^*(\mu)) > \mu$ for all $\mu \in (\mu^d, 1)$, and $p_G^*(\mu)$ and $V_\ell^*(\mu)$ are continuous and strictly increasing in $\mu \in (\mu^d, 1]$. This means that for $\mu \in (\mu^d, \bar{\mu})$, the values $y^*(\mu)$ and $p_G^*(\mu)$ are determined by the unique intersection point of the gain graph and the loss graph on the range of y such that $\pi_{Bb}(\mu, y) \geq \mu^d$.

Moreover the same argument as in equation (36) shows that for μ ($>\mu^d$) close to μ^d :

$$\begin{aligned} V_\ell^*(\pi_{Bb}(\mu, y^*(\mu))) - V_\ell^*(0) &< p_G(\pi_{Bb}(\mu, y^*(\mu)), \hat{y}) - \frac{(1-\delta)\ell}{1-\delta\ell} \\ \text{and} \quad p_G(\mu, y^*(\mu)) &= \delta(V_\ell^*(\pi_{Bb}(\mu, y^*(\mu))) - V_\ell^*(0)), \end{aligned}$$

which yield

$$p_G(\mu^d, \lim_{\mu \downarrow \mu^d} y^*(\mu)) \leq \delta\Big(p_G(\pi_{Bb}(\mu^d, \lim_{\mu \downarrow \mu^d} y^*(\mu)), \hat{y}) - \frac{(1-\delta)\ell}{1-\delta\ell}\Big) < p_G(\pi_{Bb}(\mu^d, \lim_{\mu \downarrow \mu^d} y^*(\mu)), \hat{y}).$$

These inequalities cannot hold if $\lim_{\mu \downarrow \mu^d} y^*(\mu) = \hat{y}$, because $\pi_{Bb}(\mu^d, \hat{y}) = \mu^d$. Hence, $\lim_{\mu \downarrow \mu^d} y^*(\mu) > \hat{y}$.

If $y^*(\mu) \geq \hat{y} + \varepsilon$ in a neighborhood of μ^d for some $\varepsilon > 0$, then since $\pi_{Bb}(\mu, y^*(\mu)) > \mu^d$, the same logic used to prove [A] would verify that $y^*(.)$ is continuous in a neighborhhod of μ^d, contradicting the definition of μ^d. Hence, there exist a sequence μ_n ($< \mu^d$) converging to μ^d such that $\lim_{n \to +\infty} y^*(\mu_n) \leq \hat{y} < \lim_{\mu \downarrow \mu^d} y^*(\mu)$.

In particular, $\lim_{n \to +\infty} p_G(\mu_n, y^*(\mu_n)) > \lim_{\mu \downarrow \mu^d} p_G^*(\mu)$. Since $y^*(\mu) < 1$ for $\mu = \mu_n$ and $\mu \in (\mu^d, \bar{\mu})$,

$$p_G(\mu, y^*(\mu)) = \delta(V_\ell^*(\pi_{Bb}(\mu, y^*(\mu))) - V_\ell^*(0))$$

holds for $\mu = \mu_n$ and $\mu \in (\mu^d, \bar{\mu})$. Thus, $\lim_{n\to+\infty} V_\ell^*(\pi_{Bb}(\mu_n, y^*(\mu_n))) > \lim_{\mu\downarrow\mu^d} V_\ell^*(\pi_{Bb}(\mu, y^*(\mu)))$. Given $\lim_{n\to+\infty} \pi_{Bb}(\mu_n, y^*(\mu_n)) = \pi_{Bb}(\mu^d, \lim_{n\to+\infty} y^*(\mu_n)) \leq \mu^d = \pi_{Bb}(\mu^d, \hat{y})$, we find that

$$\lim_{\mu\downarrow\mu^d} V_\ell^*(\pi_{Bb}(\mu, y^*(\mu))) < \sup_{\mu\leq\mu^d} V_\ell^*(\mu).$$

Take $\mu'' < \mu^d$ such that $V_\ell^*(\mu'')$ is arbitrarily close to $\sup_{\mu\leq\mu^d} V_\ell^*(\mu) < V_\ell^*(1)$.[21] Consider the unique $\mu^+ > \mu^d$ such that $V_\ell^*(\mu^+) = \sup_{\mu\leq\mu^d} V_\ell^*(\mu)$. Then, $\pi_{Bb}(\mu^+, y^*(\mu^+)) > \mu^+$ by Lemma 6 and, since $y^*(\mu^+) > 0$,

$$p_G^*(\mu^+) \geq \delta(V_\ell^*(\pi_{Bb}(\mu^+, y^*(\mu^+))) - V_\ell^*(0)). \tag{37}$$

We then verify that $p_G^*(\mu^+) \geq p_G^*(\mu'')$. This is immediate if $\pi_{Bb}(\mu^+, y^*(\mu^+)) = 1$ since $p_G^*(\mu'') < \delta(V_\ell^*(1) - V_\ell^*(0))$. If $\pi_{Bb}(\mu^+, y^*(\mu^+)) < 1$, the gain and loss graphs for μ'' cannot cross at any y such that $V_\ell^*(\pi_{Bb}(\mu'', y)) > V_\ell^*(\pi_{Bb}(\mu^+, y^*(\mu^+))$. To see this, note that the two graphs crossing at such y would require that $\pi_{Bb}(\mu'', y) > \pi_{Bb}(\mu^+, y^*(\mu^+))$ and thus $y > y^*(\mu^+)$, but this would imply that (37) holds with equality and, therefore, $p_G(\mu'', y) < p_G^*(\mu^+) < \delta(V_\ell^*(\pi_{Bb}(\mu'', y)) - V_\ell^*(0))$, contradicting the two graphs crossing at y. Then, $y^*(\mu'') < 1$ and thus, $p_G^*(\mu'') = \delta(V_\ell^*(\pi_{Bb}(\mu'', y^*(\mu''))) - V_\ell^*(0)) \leq p_G^*(\mu^+)$ from (37), verifying the claim.

Furthermore, $V_\ell^*(\pi_{Gg}^*(\mu^+)) > V_\ell^*(\pi_{Gg}^*(\mu''))$ because $\pi_{Gg}^*(\mu^+) > \max\{\mu^+, \pi_{Gg}^*(\mu'')\}$ and V_ℓ^* is strictly increasing in $\mu > \mu^+$ and $V_\ell^*(\mu^+) = \sup_{\mu\leq\mu^d} V_\ell^*(\mu)$. In light of (27), these observations dictate $V_\ell^*(\mu'') < V_\ell^*(\mu^+)$, contradicting to $V_\ell^*(\mu'')$ being arbitrarily close to $\sup_{\mu\leq\mu^d} V_\ell^*(\mu) = V_\ell^*(\mu^+)$. Therefore, we conclude that y^* is continuous at all $\mu \in (0, \mu_0]$. By Lemma 6, this means that V_ℓ^* is continuous and strictly increasing on $(0, 1]$.

Finally, consider the possibility that V_ℓ^* is discontinuous at $\mu = 0$, i.e., $V_\ell^*(0) < V_\ell^*(0^+) = \lim_{\mu\downarrow 0} V_\ell^*(\mu)$. If $\delta(V_\ell^*(0^+) - V_\ell^*(0)) \leq \ell = p_G(0, 1)$, then $y^*(\mu) \to 1$ as $\mu \to 0$ and thus, we would have $V_\ell^*(0^+) = \ell + \delta(\ell V_\ell^*(0^+) + (1-\ell)V_\ell^*(0)) \Rightarrow V_\ell^*(0^+) = V_\ell^*(0)$, a contradiction. If $\delta(V_\ell^*(0^+) - V_\ell^*(0)) > \ell$, then $\lim_{\mu\downarrow 0} p_G^*(\mu) = \delta(V_\ell^*(0^+) - V_\ell^*(0))$ and thus, we would have $V_\ell^*(0^+) = \delta(V_\ell^*(0^+) - V_\ell^*(0)) + \delta(\ell V_\ell^*(0^+) + (1-\ell)V_\ell^*(0)) \Rightarrow V_\ell^*(0^+) = -\delta\ell V_\ell^*(0)/(1-\delta-\delta\ell) \Rightarrow V_\ell^*(0^+) - V_\ell^*(0) = \frac{\ell}{\delta+\delta\ell-1} > \frac{\ell}{\delta\ell} > 1 > \Delta$, a contradiction to $V_\ell^*(\mu) < V_\ell^*(1)$ for all $\mu < 1$ (Lemma 3). This completes the proof. ■

Next, to characterize V_ℓ^* as a fixed point of the operator defined on $\mathcal{F}$, we extend the definition of the "pseudo-best-response" function y_V to all function $V \in \mathcal{F}$. Note that since $\bar{\mu}$ is independent of V, (9) determines $y_V(0) = y_V(\mu) = 1$ for all $\mu \geq \bar{\mu}$. For $0 < \mu < \bar{\mu}$, we extend the definition of $y_V(\mu)$ to be the unique $y \in (0, 1)$ that satisfies

$$\delta \lim_{y'\uparrow y} (V(\pi_{Bb}(\mu, y')) - V(0)) \leq p_G(\mu, y) \leq \delta\left(V(\pi_{Bb}(\mu, y)) - V(0)\right). \tag{38}$$

This uniquely determines the pseudo-best-response function y_V as

$$y_V(\mu) = \begin{cases} 1 & \text{if} \quad \mu > \bar{\mu} \\ \text{the unique } y \text{ that satisfies (38)} & \text{if} \quad 0 < \mu \leq \bar{\mu} \\ 1 & \text{if} \quad \mu = 0. \end{cases} \tag{39}$$

[21] The last inequality: For $V_\ell^*(\mu)$ to be arbitrarily close to $V_\ell^*(1)$ for some $\mu \leq \mu^d$, we need $p_G^*(\mu)$ arbitrarily close to 1 by (27), but then $y^*(\mu) = 1$ would be optimal due to $\delta\Delta < 1$, contradicting $p_G^*(\mu)$ being arbitrarily close to 1.

Lemma 7 *For any $V \in \mathcal{F}$, $y_V(\mu)$ is continuous and strictly positive on $[0,1]$ and $p_G(\mu, y_V(\mu))$ is nondecreasing in μ.*

Proof. For each $\mu \in (0, \bar{\mu}]$, by construction, $y_V(\mu)$ is the value of y at which the gain graph of $p_G(\mu, y)$ intersects with the "connected" loss graph of $\delta(V(\pi_{Bb}(\mu, y)) - V(0))$, i.e., the latter graph is connected vertically at every discontinuity points by the shortest distance. Since both of the graphs are continuous as functions of μ, the intersection point changes continuously in μ, i.e., $y_V(\mu)$ is continuous on $\mu \in (0, \bar{\mu}]$. Since $p_G(\mu, 0) = 1 > \delta\Delta$ by (1), the intersection takes place at some $y > 0$, establishing that $y_V(\mu) > 0$ for $\mu \in (0, \bar{\mu}]$ as well as when $\mu = 0$ and $\mu > \bar{\mu}$ as per (39).

Furthermore, note that $y_V(\mu) \to 1$ as $\mu \to 0$ because, for every $y < 1$, $\pi_{Bb}(\mu, y) \to 0$ as $\mu \to 0$ and thus, $\delta(V(\pi_{Bb}(\mu, y)) - V(0)) < \ell \leq p_G(\mu, y)$ for all μ sufficiently small. Since $y_V(\bar{\mu}) = 1$ by construction (using $y_V(0) = 1$ if $\bar{\mu} = 0$), it follows that $y_V(\mu)$ is continuous on $[0, 1]$.

For $\mu \geq \bar{\mu}$, we have $p_G(\mu, y_V(\mu)) = p_G(\mu, 1)$ which increases in μ by (1). For $\mu \in (0, \bar{\mu})$, the two aforementioned graphs move upward as μ increases because both $p_G(\mu, y)$ and $\pi_{Bb}(\mu, y)$ increases in μ by (1) and (3), respectively. Hence, the height of the intersection point also increases, i.e., $p_G(\mu, y_V(\mu))$ weakly increases in μ. ■

Since $\pi^*_{Gg}(\mu)$ increases in μ, $T(V)$ as defined in (11) is non-decreasing and right-continuous in μ by Lemma 7. In addition, $T(V)(0) = p_G(0, 1) + \delta(\ell V(0) + (1-\ell)V(0))$ which yields $T(V)(0) = \ell/(1-\delta)$; and $T(V)(1) = 1 + \delta(\ell V(1) + (1-\ell)V(0))$ which yields $T(V)(1) = \ell/(1-\delta) + \Delta$. Hence, the operator T is well-defined on $\mathcal{F}$ by (11).

Lemma 8 *In any honest equilibrium, V^*_ℓ is a fixed point of T and $y^*(\mu) = y_{V^*_\ell}(\mu)$.*

Proof. For any equilibrium value function V^*_ℓ, (8) and (39) imply that $y^*(\mu) = y_{V^*_\ell}(\mu)$ for all μ. Since $y^*(\mu) = y_{V^*_\ell}(\mu) > 0$ by Lemma 7, we deduce from (7), (8) and (11) that $T(V^*_\ell)(\mu) = V^*_\ell(\mu)$ for all μ. ■

By combining Lemmas 1–8, at this point we have proved all claims of Proposition 1 except the existence and uniqueness of the fixed point, which we now turn to.

Proof of existence. Endowed with the topology of the weak convergence, the set $\mathcal{F}$ is convex and compact (Theorem 5.1, Billingsley, 1999). By Fan-Glicksberg Fixed Point Theorem,[22] therefore, T has a fixed point in $\mathcal{F}$ if T is continuous on $\mathcal{F}$, which we show below.

Consider a sequence V_n, $n = 1, 2, \cdots$, in $\mathcal{F}$ that weakly converges to $V \in \mathcal{F}$. To prove continuity of T, we show below that $T(V_n)$ weakly converges to $T(V)$, i.e., $T(V_n)(\mu)$ converges to $T(V)(\mu)$ at all continuity points of $T(V)$ (Theorem 2.1, Billingsley, 1999).

Let Ω be the set of all points where $V(\pi^*_{Gg}(\mu))$ is continuous. Since $\pi^*_{Gg}(\mu)$ is increasing, $[0,1]\backslash\Omega$ is countable. Since V is continuous at $\pi^*_{Gg}(\mu)$ if $\mu \in \Omega$ by continuity of π^*_{Gg}, weak convergence of V_n implies that $V_n(\pi^*_{Gg}(\mu))$ converges to $V(\pi^*_{Gg}(\mu))$ on Ω.

[22] This theorem (Fan, 1952; Glicksberg, 1952) states that an upper hemi-continuous convex valued correspondence from a nonempty compact convex subset of a convex Hausdorff topological vector space has a fixed point.

Next, Let $y_V(\mu)$ be as defined in (39) for V and $y_{V_n}(\mu)$ for V_n. Let Λ be the set of points where $V(\pi_{Bb}(\mu, y_V(\mu)))$ is continuous. Since $\pi_{Bb}(\mu, y_V(\mu))$ is non-decreasing on $(0,1]$ as verified in the proof of Lemma 7, $[0,1]\backslash\Lambda$ is countable. We now show that $y_{V_n}(\mu) \to y_V(\mu)$ for all $\mu \in \Lambda$.

Consider $\mu \in \Lambda$. That $y_{V_n}(\mu) \to y_V(\mu)$ is trivial from (39) if $\mu = 0$ or $\mu > \bar{\mu}$. Hence, suppose $0 < \mu \leq \bar{\mu}$ so that, denoting $V_n^-(\pi_{Bb}(\mu, y)) = \lim_{x \uparrow y} V_n(\pi_{Bb}(\mu, x))$, we have

$$\delta\big(V_n^-(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)\big) \leq p_G(\mu, y_{V_n}(\mu)) \leq \delta\big(V_n(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)\big). \quad (40)$$

By taking a subsequence if necessary, we may assume that $y_{V_n}(\mu)$ converges to a limit y'. To reach a contradiction, suppose $y' \neq y_V(\mu)$. First, consider the case that $y' < y_V(\mu)$. Then, since $p_G(\mu, y)$ decreases with μ there exists $\varepsilon > 0$ such that

$$p_G(\mu, y_{V_n}(\mu)) > p_G(\mu, y_V(\mu)) + \varepsilon = \delta\big(V(\pi_{Bb}(\mu, y_V(\mu))) - V(0)\big) + \varepsilon$$

for sufficiently large n, where the equality follows because $\mu \in \Lambda$. From this we further deduce that

$$\begin{aligned} p_G(\mu, y_{V_n}(\mu)) &> \delta(V_n(\pi_{Bb}(\mu, y_V(\mu))) - V(0)) + \varepsilon/2 \\ &> \delta(V_n(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)) + \varepsilon/2 \end{aligned}$$

for sufficiently large n, where the first inequality follows because $V_n(\pi_{Bb}(\mu, y_V(\mu))) \to V(\pi_{Bb}(\mu, y_V(\mu)))$ for $\mu \in \Lambda$ and the second because $\pi_{Bb}(\mu, y)$ increases in y and $y_{V_n}(\mu) \to y' < y_V(\mu)$. However, this contradicts (40).

For the case $y' > y_V(\mu)$, we can apply the same reasoning using $V_n^-(\pi_{Bb}(\mu, y_{V_n}(\mu))) \geq V_n(\pi_{Bb}(\mu, y_V(\mu)))$ for n large to reach an analogous contradiction:

$$p_G(\mu, y_{V_n}(\mu)) < \delta\big(V_n^-(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)\big) - \varepsilon/2.$$

Hence, we conclude that $y_{V_n}(\mu) \to y_V(\mu)$ for all $\mu \in \Lambda$.

Together with the earlier result that $V_n(\pi^*_{Gg}(\mu)) \to V(\pi^*_{Gg}(\mu))$ for all $\mu \in \Omega$, this establishes for all $\mu \in \Omega \cap \Lambda$ that

$$\begin{aligned} T(V_n)(\mu) &= p_G(\mu, y_{V_n}(\mu)) + \delta\big(\ell V_n(\pi^*_{Gg}(\mu)) + (1-\ell)V(0)\big) \\ &\to p_G(\mu, y_V(\mu)) + \delta\big(\ell V(\pi^*_{Gg}(\mu)) + (1-\ell)V(0)\big) = T(V)(\mu) \end{aligned}$$

as $n \to \infty$. Finally, to verify this convergence at every continuity point of $T(V)(\mu)$, observe first that this convergence is trivial from (39) at $\mu = 0, 1$. For any other $\mu \notin \Omega \cap \Lambda$ at which $T(V)$ is continuous, one can find $\mu_1 \in \Omega \cap \Lambda \cap (0, \mu)$ arbitrarily close to μ and $\mu_2 \in \Omega \cap \Lambda \cap (\mu, 1)$ arbitrarily close to μ because $\Omega \cap \Lambda$ is dense in $[0,1]$. Since $T(V_n)(\mu_1) \leq T(V_n)(\mu) \leq T(V_n)(\mu_2)$ and $T(V)(\mu_1) \leq T(V)(\mu) \leq T(V)(\mu_2)$, taking the limits we get

$$T(V)(\mu_1) \leq \liminf T(V_n)(\mu) \leq \limsup T(V_n)(\mu) \leq T(V)(\mu_2), \quad \text{and}$$
$$\sup_{\substack{\mu_1 \in \Omega\cap\Lambda \\ \mu_1 < \mu}} T(V)(\mu_1) = T(V)(\mu) = \inf_{\substack{\mu_2 \in \Omega\cap\Lambda \\ \mu_2 > \mu}} T(V)(\mu_2),$$

which imply, as desired, that $T(V_n)(\mu)$ converges to $T(V)(\mu)$ at every continuity point of $T(V)(\mu)$. This proves that T is continuous and thus, completes the proof of existence.

Proof of uniqueness. To reach a contradiction, suppose there are two fixed points V^1 and V^2. Notice that V^1 and V^2 are continuous by Lemma 1 and

$$V^i(\mu) = p_G(\mu, 1) + \delta\left(\ell V(1) + (1-\ell)V(0)\right) \quad \forall \mu \geq \bar{\mu}, \quad i = 1, 2, \tag{41}$$

in particular, $V^1(\mu) = V^2(\mu)$ for all $\mu \geq \bar{\mu}$. Thus, the following is well-defined:

$$\hat{\mu} := \min\{\mu \,|\, V^1(\mu') = V^2(\mu') \ \forall \mu' \geq \mu\} \in (0, \bar{\mu}]. \tag{42}$$

A "segment" for $i = 1, 2$, is a nonempty interval $I_i = [x, z] \subset [0, \bar{\mu}]$ such that $V^i(\mu) > V^j(\mu)$ for all $\mu \in (x, z)$ and $V^i(\mu) = V^j(\mu)$ for $\mu = x, z$, where $j \neq i$. A "region" for $i = 1, 2$, is a nonempty interval $R_i = [x, z] \subset [0, \bar{\mu}]$ such that $V^i(\mu) \geq V^j(\mu)$ for all $\mu \in I_i$ and there are $x', z' \in R_i$ such that $[x, x']$ and $[z', z]$ are segments for i. Let

$$p_G^i(\mu) := p_G(\mu, y_{V^i}(\mu)) \quad \text{and} \quad \pi_{Bb}^i(\mu) := \pi_{Bb}(\mu, y_{V^i}(\mu)) \quad \text{for} \quad i = 1, 2. \tag{43}$$

Recall that in the proof of Lemma 7, we have shown that both $p_G^i(\mu)$ and $\pi_{Bb}^i(\mu)$ weakly increase in μ. Since V^i strictly increases in μ by Lemma 1, the same reasoning establishes that

[B] $p_G^i(\mu)$ and $\pi_{Bb}^i(\mu)$ strictly increase in μ.

Next, we establish the following:

[C] If $V^1(\pi_{Bb}^i(\mu)) = V^2(\pi_{Bb}^i(\mu))$ for some $\mu > 0$ and some $i = 1, 2$, then $y_{V^1}(\mu) = y_{V^2}(\mu)$ and consequently, $p_G^1(\mu) = p_G^2(\mu)$ and $\pi_{Bb}^1(\mu) = \pi_{Bb}^2(\mu)$. If, in addition, $V^1(\pi_{Gg}^*(\mu)) = V^2(\pi_{Gg}^*(\mu))$ holds, then $V^1(\mu) = V^2(\mu)$.

Note that this observation is trivial for $\mu \geq \bar{\mu}$. Since

$$p_G^i(\mu) = \delta\big(V^i(\pi_{Bb}^i(\mu)) - V^i(0)\big) \quad \forall \mu \in (0, \bar{\mu}], \tag{44}$$

$V^1(\pi_{Bb}^i(\mu)) = V^2(\pi_{Bb}^i(\mu))$ implies $p_G^1(\mu) = p_G^2(\mu)$, which in turn implies $y_{V^1}(\mu) = y_{V^2}(\mu)$, from which the remaining claims of [C] follow.

Finally, since $\pi_{Bb}^i(\hat{\mu}) > \hat{\mu}$ by Lemma 6 and $\pi_{Gg}^*(\hat{\mu}) > \hat{\mu}$ by (4), due to continuity, there is $\mu' < \hat{\mu}$ such that $V^1(\mu') \neq V^2(\mu')$, $\pi_{Bb}^i(\mu') > \hat{\mu}$ and $\pi_{Gg}^*(\mu') > \hat{\mu}$. Then, $V^1(\pi_{Bb}^i(\mu')) = V^2(\pi_{Bb}^i(\mu'))$ by (42) and thus, $V^1(\mu') = V^2(\mu')$ by [C], a contradiction to the earlier assertion that $V^1(\mu') \neq V^2(\mu')$. This completes the proof of uniqueness, hence the proof of Proposition 1.

B. Other Proofs

Proof of Theorem 1. By construction, optimality of ℓ-seller strategy is satisfies for $y^* = y_{V_\ell^*}$ at all $\mu > 0$ where V_ℓ^* is the unique fixed point of T. At $\mu = 0$, it is satisfied if $\pi_{Bb}^*(0)$, which is undefined by Bayes rule, is not too high, e.g., when $\pi_{Bb}^*(0) = 0$.

Recall V_h^* is h-seller's value function given the price schedule p_m^* and transition rule π_{mq}^*, which is calculated as

$$V_h^*(\mu) = \sum_{t=0}^{\infty} \sum_{\mathbf{h}^t \in H_g^t} \delta^t \rho(\mathbf{h}^t) p_G\big(\pi(\mathbf{h}^t, \mu), y^*(\pi(\mathbf{h}^t, \mu))\big) \quad \text{for} \quad \mu > 0 \tag{45}$$

where $H_g^t := \{g, b\}^{t-1} \times \{g\}$ is the set of all possible realizations of q for t periods with the requirement that $q = g$ in period t; $\rho(\mathbf{h}^t)$ is the ex ante probability that $\mathbf{h}^t \in H_g^t$ realizes; $\pi(\mathbf{h}^t, \mu)$ is the posterior belief at the beginning of period t calculated by Bayes rule from the prior belief μ along $\mathbf{h}^t$. Observe that $V_h^*(\mu)$ is increasing in μ because $p_G(\mu, y^*(\mu))$, $\pi_{Gg}^*(\mu)$ and $\pi_{Bb}(\mu, y^*(\mu))$ all increase in μ as verified earlier.

Since $p_G^*(\mu) > p_B^*(\mu)$ and $\pi_{Bg}^*(\mu) = 0$, upon drawing $q = g$ it is clearly optimal for an h-seller is to announce $m = G$ truthfully. It remains to show optimality of truthful announcement upon drawing $q = b$.

For $\mu \in [\bar{\mu}, 1]$, this follows from (13) because $\pi_{Bb}(\mu, y^*(\mu)) = 1$ and $p_G(\mu, y^*(\mu)) \leq 1$. For $\mu \in (0, \bar{\mu})$, observe from (38) and (39) that

$$\delta\big(V_\ell^*(\pi_{Bb}^*(\mu)) - V_\ell^*(0)\big) = p_G^*(\mu). \tag{46}$$

Since $V_\ell^*(0) = V_h^*(0)$ from (12), while

$$V_h^*(\mu) > V_\ell^*(\mu) \quad \forall \mu > 0 \tag{47}$$

as will be verified below, it follows that

$$\delta\big(V_h^*(\pi_{Bb}^*(\mu)) - V_h^*(0)\big) > p_G^*(\mu). \tag{48}$$

This proves optimality of truthful announcement upon drawing $q = b$ for $\mu > 0$.

We already verified that lying when $\mu = 0$ and $q = b$ is optimal for h-seller with $\pi_{Bb}^*(0) = 0$, which completes verification of an honest equilibrium. Other honest equilibria may exist that differ in what h-seller does when $\mu = 0$ and $q = b$. But since consistency requires that an h-seller starts with an initial reputation level $\mu > 0$ and an h-seller always tells the truth as per Condition H, the difference pertains to off-equilibrium path. Therefore, the equilibrium outcome is unique.

Finally, we prove (47). Let $V_h(\mu)$ be the value function from the following strategy of an h-seller: always report $q = g$ truthfully and upon drawing $q = b$ for the first time report $m = G$ and get $V_h^*(0)$ in the continuation subgame. Then,

$$V_h(\mu) = \Big[\sum_{t=0}^{\infty} h^t \delta^t p_G(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu)))\Big] + \delta V_h^*(0)(1-h) \sum_{t=1}^{\infty} h^t \delta^t \tag{49}$$

where $\pi_{Gg}^t(\mu)$ is as defined in (28). Since $V_h^*(\mu) \geq V_h(\mu)$ is clear from definition of V_h^*, it suffices to show $V_h(\mu) - V_\ell^*(\mu) > 0$. Subtracting (29) from (49),

$$V_h(\mu) - V_\ell^*(\mu) = \Big[\sum_{t=0}^{\infty} (h^t - \ell^t) \delta^t p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big)\Big] + \delta\Big(\frac{1-h}{1-\delta h} - \frac{1-\ell}{1-\delta \ell}\Big) V_\ell^*(0).$$

Since $p_G\big(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))\big) > \ell$ for $\mu > 0$, (47) follows from

$$V_h(\mu) - V^*_\ell(\mu) > \frac{\delta(h-\ell)\ell}{(1-\delta h)(1-\delta\ell)} - \frac{\delta(1-\delta)(h-\ell)}{(1-\delta h)(1-\delta\ell)} V^*_\ell(0) = 0.$$

This completes proof of Theorem 1. For later use, however, we also prove the following nested result:

[S] If $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$ and δ is large enough, there exists an honest equilibrium in which h-seller announces truthfully even when $\mu = 0$.

To prove this, let

$$V^o_h(\mu) := h\sum_{t=0}^{\infty} h^t\delta^t p_G\big(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))\big) \quad \forall\mu > 0$$

so that

$$V^*_h(\mu) = V^o_h(\mu) + (1-h)\delta\sum_{t=0}^{\infty} h^t\delta^t V^*_h\big(\pi_{Bb}(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))\big) \quad \forall\mu > 0. \tag{50}$$

In conjunction with (29), we have

$$V^o_h(\mu) - V^*_\ell(\mu) = \Big[\sum_{t=0}^{\infty}(h^{t+1}-\ell^t)\delta^t p_G\big(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))\big)\Big] - \delta V^*_\ell(0)\frac{1-\ell}{1-\delta\ell}.$$

For $\mu \geq \bar{\mu}$, since $y^*(\mu) = 1$ we have

$$\begin{aligned}
\frac{dV^o_h(\mu)}{d\mu} - \frac{dV^*_\ell(\mu)}{d\mu} &= \sum_{t=0}^{\infty}(h^{t+1}-\ell^t)\delta^t \frac{\partial p_G\big(\pi^t_{Gg}(\mu),1\big)}{\partial\mu}\frac{d\pi^t_{Gg}(\mu)}{d\mu} &(51)\\
&= \sum_{t=0}^{\infty}\delta^{2t}\Bigg[(h^{2t+1}-\ell^{2t})\frac{\partial p_G\big(\pi^{2t}_{Gg}(\mu),1\big)}{\partial\mu}\frac{d\pi^{2t}_{Gg}(\mu)}{d\mu}\\
&\qquad + \delta(h^{2t+2}-\ell^{2t+1})\frac{\partial p_G\big(\pi^{2t+1}_{Gg}(\mu),1\big)}{\partial\mu}\frac{d\pi^{2t+1}_{Gg}(\mu)}{d\mu}\Bigg]\\
&> \sum_{t=0}^{\infty}\delta^{2t}\ell^{2t}\Bigg[(h-1)\frac{\partial p_G\big(\pi^{2t}_{Gg}(\mu),1\big)}{\partial\mu}\frac{d\pi^{2t}_{Gg}(\mu)}{d\mu}\\
&\qquad + \delta(h^2-\ell)\frac{\partial p_G\big(\pi^{2t+1}_{Gg}(\mu),1\big)}{\partial\mu}\frac{d\pi^*_{Gg}(\pi^{2t}_{Gg}(\mu))}{d\mu}\frac{d\pi^{2t}_{Gg}(\mu)}{d\mu}\Bigg]. &(52)
\end{aligned}$$

By routine calculation, we get

$$\begin{aligned}
&(h-1)\frac{\partial p_G\big(\mu,1\big)}{\partial\mu} + (h^2-\ell)\frac{\partial p_G\big(\pi^*_{Gg}(\mu),1\big)}{\partial\mu}\frac{d\pi^*_{Gg}(\mu)}{d\mu}\\
&\qquad = -\frac{h(1-h)(1-\ell)}{(1-(1-h)\mu)^2} + \frac{h^2(h^2-\ell)(1-\ell)\ell}{(\ell(1-\mu)+h^2\mu)^2}, &(53)
\end{aligned}$$

the derivative of which is

$$-2(1-\ell)\frac{h(1-h)^2}{(1-(1-h)\mu)^3}-\frac{\ell(h^3-h\ell)^2}{(\ell(1-\mu)+h^2\mu)^3}<0. \tag{54}$$

If $h>\frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$, it is routinely verified that (53) evaluated at $\mu=1$ is positive and thus, (53) is positive for all μ due to (54). This further implies that (52) is positive for all $\mu\geq\bar{\mu}$ and consequently, from (50),

$$\frac{dV_h^*(\mu)}{d\mu}\geq\frac{dV_\ell^*(\mu)}{d\mu}\quad\forall\mu\geq\bar{\mu} \tag{55}$$

when $\delta<1$ is sufficiently close to 1, provided that $h>\frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$.

Next, let $\mu_m=\min\{\mu|\pi_{Gg}^*(\mu)\geq\bar{\mu}$ and $\pi_{Bb}(\mu,y^*(\mu))\geq\bar{\mu}\}$ and consider $\mu\in[\mu_m,\bar{\mu}]$. Note that $\mu_m<\bar{\mu}$ due to Lemmas 1 and 6. Since

$$\begin{aligned}V_h^*(\mu) &= hp_G(\mu,y^*(\mu))+\delta\big(hV_h^*(\pi_{Gg}^*(\mu))+(1-h)V_h^*(\pi_{Bb}(\mu,y^*(\mu)))\big) \text{ and}\\ V_\ell^*(\mu) &= p_G(\mu,y^*(\mu))+\delta\big(\ell V_\ell^*(\pi_{Gg}^*(\mu))+(1-\ell)V_\ell^*(0)\big),\end{aligned}$$

we deduce that $\frac{dV_h^*(\mu)}{d\mu}-\frac{dV_\ell^*(\mu)}{d\mu}$, which exists almost everywhere because both $V_h^*(\mu)$ and $V_\ell^*(\mu)$ are continuous and increasing, is equal to the derivative of

$$(1-h)\big(\delta V_h^*(\pi_{Bb}(\mu,y^*(\mu)))-p_G(\mu,y^*(\mu))\big)+\delta\big(hV_h^*(\pi_{Gg}^*(\mu))-\ell V_\ell^*(\pi_{Gg}^*(\mu))\big),$$

which is positive due to (55) because $p_G(\mu,y^*(\mu))=\delta\big(V_\ell^*(\pi_{Bb}(\mu,y^*(\mu)))-V_\ell^*(0)\big)$ for $\mu\leq\bar{\mu}$. Repeated application of analogous argument establishes that $\frac{dV_h^*(\mu)}{d\mu}>\frac{dV_\ell^*(\mu)}{d\mu}$ for all $\mu>0$ when $\delta<1$ is sufficiently close to 1 if $h>\frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$.

Setting $\pi_{Bb}(0,1)=\lim_{\mu\to0}\pi_{Bb}(\mu,y^*(\mu))$ and $V_h^*(0)=\lim_{\mu\to0}V_h^*(\mu)$, this implies that h-seller prefers to tell the truth upon drawing $q=b$ whenever ℓ-seller is indifferent, i.e., when $\mu\in(0,\bar{\mu}]$. Then, truth-telling is optimal when $\mu=0$ and $q=b$ by continuity of V_h^*, $p_G(\mu,y^*(\mu))$, $\pi_{Gg}^*(\mu)$, and $\pi_{Bb}(\mu,y^*(\mu))$. Finally, optimality of truth-telling when $q=g$ is immediate from $p_G^*(\mu)>p_B^*(\mu)$ and $\pi_{Bg}^*(\mu)=0$. ■

Proof of Properties 2–7. Properties 2 and 7 are already proved in the main text. For Property 3, it only remains to prove (15). This is done in the proof of Lemma 6 which also proves Property 6. Property 4 is proved by applying the argument in the proof of Lemma 7 of verifying monotonicity of $p_G(\mu,y_V(\mu))$ to V_ℓ^* which is continuous and strictly increasing by Lemma 1.

We now prove Property 5. It is clear from (6), (1) and (10) that $\bar{\mu}$ strictly increases in δ. Consider $0<\delta<\delta'<1$ and let $y^*(\cdot|\delta)$ and $y^*(\cdot|\delta')$ denote $y^*(\cdot)$ for different δ and similarly for other equilibrium variables. To reach a contradiction, suppose that $y^*(\mu|\delta)\leq y^*(\mu|\delta')$ for some $\mu\in(0,\bar{\mu})$ where $\bar{\mu}$ is associated with δ. Then, $\mu'=\max\{\mu<\bar{\mu}\,|\,y^*(\mu|\delta)\leq y^*(\mu|\delta')\}$ is well-defined. Note that $y^*(\mu'|\delta)=y^*(\mu'|\delta')<1$ and thus, $p_G^*(\mu'|\delta)=p_G^*(\mu'|\delta')$ and $\delta(V_\ell^*(\mu'|\delta)-V_\ell^*(0))=\delta'(V_\ell^*(\mu'|\delta')-V_\ell^*(0))$. However, since $y^*(\mu|\delta)\geq y^*(\mu|\delta')$ for all

$\mu \geq \mu'$, from (30) we derive a contradiction:

$$\delta(V_\ell^*(\mu'|\delta) - V_\ell^*(0)) = \sum_{t=0}^{\infty} \delta^{t+1}\ell^t \big(p_G(\pi_{Gg}^t(\mu'), y^*(\pi_{Gg}^t(\mu')|\delta)) - \ell\big)$$
$$< \sum_{t=0}^{\infty}(\delta')^{t+1}\ell^t \big(p_G(\pi_{Gg}^t(\mu'), y^*(\pi_{Gg}^t(\mu')|\delta')) - \ell\big) = \delta'(V_\ell^*(\mu'|\delta') - V_\ell^*(0)).$$

This completes the proof. ■

Proof of Lemma 2. Consider an honest equilibrium in which a seller never lies upon drawing a good quality item, so that it is characterized by the probability $y^\dagger(\mu)$ that an ℓ-seller lies when $q = b$ and the value functions $V_\theta^\dagger$ for $\theta = h, \ell$. Adopting the convention, for the same reason as before, that an ℓ-seller always announce G when $\mu = 0$, we have $V_\ell^\dagger(0) = p_G^*(0) + \delta \max\{v_o, V_\ell^\dagger(0)\} = \ell + \delta v_o > \frac{\ell}{1-\delta}$; and, analogously to (24), $V_\ell^\dagger(\mu) > V_\ell^\dagger(0)$ is immediate for all $\mu > 0$. In addition, an argument analogous to the proof of Lemma 4 establishes that without loss of generality we may set $\pi_{Gb}^\dagger(1) = 0$ and $V_\ell^\dagger(1) = \frac{1+\delta(1-\ell)v_o}{(1-\delta\ell)}$. Thus, in particular, we have verified (16). Consequently, the relevant steps within the proof of Proposition 1 extend straightforwardly to verify that $V_\ell^\dagger$ is continuous and strictly increasing in μ.

Define $\mathcal{F}_{v_o}$ to be the set of all non-decreasing and right-continuous functions V on $[0, 1]$ such that $V(0) = V_\ell^\dagger(0)$ and $V(1) = V_\ell^\dagger(1)$. Define $y_V^\dagger(\mu)$ in the same manner as in (38) and (39) with $V(0)$ replaced by v_o and $\bar{\mu}$ replaced by $\bar{\mu}^\dagger := \inf\{\mu \mid p_G(\mu, 1) > \delta\Delta_{v_o}\} < \bar{\mu}$ where the last inequality follows from $\Delta_{v_o} < \Delta$. As long as $\delta\Delta_{v_o} > \ell$ so that $y_V^\dagger(\mu) < 1$ for some μ, which we assume below (else, $y_V^\dagger(\mu) \equiv 1$ and the proof is simpler), we have $y_V^\dagger(\mu) \in (0, 1)$ for $\mu \in (0, \bar{\mu}^\dagger)$ with $\lim_{\mu\to 0} y_V^\dagger(\mu) = 1$ because $\delta\big(\max\{V(\pi_{Bb}(\mu, y)), v_o\} - v_o\big)$ approaches $\delta\Delta_{v_o} > \ell$ as $y \to 1$ while it approaches 0 as $\mu \to 0$ for all $y < 1$.[23] Furthermore, $y_V^\dagger(\mu)$ is clearly continuous and assumes 1 for $\mu \geq \bar{\mu}^\dagger$. Define $T_{v_o} : \mathcal{F}_{v_o} \to \mathcal{F}_{v_o}$ as

$$T_{v_o}(V)(\mu) := p_G(\mu, y_V^\dagger(\mu)) + \delta\big(\ell \max\{v_o, V(\pi_{Gg}^*(\mu))\} + (1-\ell)v_o\big). \tag{56}$$

It is straightforward to verify that $T_{v_o}(V) \in \mathcal{F}_{v_o}$.

Then, the proof of Proposition 1 extend to T_{v_o}, establishing that, for any $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, there is a unique fixed point of T_{v_o} and it is continuous and strictly increasing. We omit the proofs because they are analogous with straightforward changes due to the seller opting to restart whenever his reputation level is so low that the continuation value falls short of v_o.[24]

Since the outside option value is v_o for an h-seller as well, optimality of truth-telling for h-seller can be verified by an argument analogous to that leading to Theorem 1, with

[23]Note that this implies $V(\pi_{Bb}(\mu, y_V^\dagger(\mu))) - v_o > p_G(\mu, y_V^\dagger(\mu))$ for all $\mu \in (0, \bar{\mu}^\dagger)$. Thus, an h-seller with any reputation $\mu > 0$ does not exit after trading a bad quality item because the value of updated reputation exceeds v_o as this inequality shows. However, both types of seller may exit after trading a good quality item in the initial period if the value of the updated reputation, $V_\theta^\dagger(\pi_{Gg}^*(\mu^i))$, falls short of v_o.

[24]In the proof of lemma 6, (30) becomes $V_\ell^\dagger(\mu) = \sum_{t=0}^\infty \delta^t\ell^t(p_G(\pi_{Gg}^t(\mu), y_V(\pi_{Gg}^t(\mu))) - \ell) + \frac{\delta v_o(1-\ell)+\ell}{1-\delta\ell}$ and thus, (35) becomes $V_\ell^\dagger(\tilde{\mu}) - v_o < \sum_{t=0}^\infty (p_G(\pi_{Gg}^t(\tilde{\mu}), \hat{y}) - \ell)\delta^t\ell^t$ because $\frac{\delta v_o(1-\ell)+\ell}{1-\delta\ell} < v_o$.

δ_h replaced by the threshold δ_{v_o} that solves

$$\delta_{v_o}\Big(\frac{h}{1-\delta_{v_o}} - v_o\Big) = 1.$$

Thus, we have shown that an honest equilibrium exists if $\delta > \delta_{v_o}$, i.e., if $\delta(\frac{h}{1-\delta} - v_o) > 1$, when sellers can exit for an outside option value v_o. ■

Proof of Proposition 2. It is immediate from the definition of $\bar{\mu}^\dagger$ that $y^\dagger(\mu) = 1$ for all $\mu \geq \bar{\mu}^\dagger$. Hence, we consider $\mu < \bar{\mu}^\dagger$ ($< \bar{\mu}$) below.

It is straightforward to extend the relevant arguments in the proof of Proposition 1, to verify that $y^\dagger$ is continuous and $y^\dagger(\mu) \in (0,1)$ for $\mu < \bar{\mu}^\dagger$. To reach a contradiction, suppose $y^\dagger(\mu') = y^*(\mu')$ for some $\mu' < \bar{\mu}^\dagger$ and $y^\dagger(\mu) > y^*(\mu)$ for all $\mu \in (\mu', \bar{\mu})$. Then,

$$\begin{aligned}\delta\big(V_\ell^*(\pi_{Bb}(\mu', y^*(\mu'))) - V_\ell^*(0)\big) &= p_G(\mu', y^*(\mu')) \\ &= p_G(\mu', y^\dagger(\mu')) = \delta\big(V_\ell^\dagger(\pi_{Bb}(\mu', y^\dagger(\mu'))) - v_o\big)\end{aligned}$$

and thus,

$$V_\ell^*(\tilde{\mu}) - V_\ell^*(0) = V_\ell^\dagger(\tilde{\mu}) - v_o \quad \text{where} \quad \tilde{\mu} := \pi_{Bb}(\mu', y^*(\mu')) > \mu' \tag{57}$$

and the inequality is from Lemma 6. Furthermore, since

$$V_\ell^*(\tilde{\mu}) = p_G(\tilde{\mu}, y^*(\tilde{\mu})) + \delta\big(\ell V_\ell^*(\pi_{Gg}^*(\tilde{\mu})) + (1-\ell)V_\ell^*(0)\big) \quad \text{and} \tag{58}$$

$$V_\ell^\dagger(\tilde{\mu}) = p_G(\tilde{\mu}, y^\dagger(\tilde{\mu})) + \delta\big(\ell V_\ell^\dagger(\pi_{Gg}^*(\tilde{\mu})) + (1-\ell)v_o\big) \tag{59}$$

while $p_G(\tilde{\mu}, y^*(\tilde{\mu})) \geq p_G(\tilde{\mu}, y^\dagger(\tilde{\mu}))$, (57)-(59) would imply

$$\delta\ell\big[\big(V_\ell^*(\pi_{Gg}^*(\tilde{\mu})) - V_\ell^*(0)\big) - \big(V_\ell^\dagger(\pi_{Gg}^*(\tilde{\mu})) - v_o\big)\big] \leq (\delta - 1)\big(v_o - V_\ell^*(0)\big) < 0. \tag{60}$$

Since $V_\ell^*(1) - V_\ell^*(0) = \Delta > \Delta_{v_o} = V_\ell^\dagger(1) - v_o$, there must exist $\mu'' \in (\tilde{\mu}, 1)$ such that $V_\ell^*(\mu'') - V_\ell^*(0) \leq V_\ell^\dagger(\mu'') - v_o$ and $V_\ell^*(\mu) - V_\ell^*(0) > V_\ell^\dagger(\mu') - v_o$ for all $\mu > \mu''$. However, since $p_G(\mu'', y^*(\mu'')) \geq p_G(\mu'', y^\dagger(\mu''))$ and $\pi_{Gg}^*(\mu'') > \mu''$, (58) and (59) evaluated at $\mu = \mu''$ imply that $V_\ell^*(\mu'') - \delta V_\ell^*(0) > V_\ell^\dagger(\mu'') - \delta v_o$ and consequently, $V_\ell^*(\mu'') - V_\ell^*(0) > V_\ell^\dagger(\mu'') - v_o$, contradicting the definition of μ''. ■

Proof of Theorem 3. Recall that μ_1 and χ_1 denote the default reputation level and stationary mass of new sellers, respectively; and v_1, $y_{v_1}^\dagger$ and $V_{v_1}^\dagger$ denote ℓ-seller's default value, strategy and value function, respectively, in equilibrium.

Let $\rho_\theta(q)$ denote the probability that a seller of type θ draws $q \in \{g, b\}$, i.e., $\rho_\theta(g) = \theta = 1 - \rho_\theta(b)$. For any k-period quality history $\mathbf{h}^k = (q_1, \cdots, q_k) \in H^k := \{g, b\}^k$, let $\rho_\theta(\mathbf{h}^k)$ be the ex-ante probability that $\mathbf{h}^k$ realizes for a seller of type θ. We use $\mathbf{h}_j^k = (q_1, \cdots, q_j)$ to denote the first j-entry truncation of $\mathbf{h}^k$.

Given a default reputation $\mu_1 > 0$, let $\pi(\mathbf{h}_j^k)$ denote the posterior reputation for a seller who has survived the history $\mathbf{h}_j^k$ without cheating, updated according to $y_{v_1}^\dagger$. Setting $\pi(\mathbf{h}_0^k) = \mu_1$, we can define $\pi(\mathbf{h}_j^k)$ recursively by:

$$\pi(\mathbf{h}_j^k) = \frac{\pi(\mathbf{h}_{j-1}^k)\rho_h(q_j)}{\pi(\mathbf{h}_{j-1}^k)\rho_h(q_j) + (1 - \pi(\mathbf{h}_{j-1}^k))\rho_\ell(q_j)(1 - y_{v_1}^\dagger(\pi(\mathbf{h}_{j-1}^k), q_j))}, \tag{61}$$

where $y^{\dagger}_{v_1}(\mu, g) = 0$ and $y^{\dagger}_{v_1}(\mu, b) = y^{\dagger}_{v_1}(\mu)$ for all μ. Then, the ex-ante probability that an ℓ-seller remains in the market without having cheated after k-period history $\mathbf{h}^k$ is

$$\Pr(\mathbf{h}^k) = \prod_{j=1}^{k} [\rho_\ell(q_j)(1 - y^{\dagger}_{v_1}(\pi(\mathbf{h}^k_{j-1}), q_j))(1 - \chi)]. \tag{62}$$

Consequently, in a stationary state, the measure of nominally k-period old ℓ-sellers who restart in period $k+1$ for $k \geq 1$, is

$$\chi_1(1 - \mu_1)\Big(\sum_{\mathbf{h}^k \in H^k} \Pr(\mathbf{h}^k)(1 - \ell) y^{\dagger}_{v_1}(\pi(\mathbf{h}^k))(1 - \chi)\Big).$$

This implies that the total measure of old ℓ-sellers who restart in an arbitrary period is $\chi_1(1 - \mu_1)\Lambda(v_1)$ where $\Lambda(v_1)$ is as defined earlier in (18).

Now, as verified in the discussion (in the main text) preceding Theorem 3, the value of v_1 in a stationary equilibrium is a fixed point that satisfies $v_1 = V^{\dagger}_{v_1}(\mu^{\dagger}_1(v_1))$ where $\mu^{\dagger}_1(v_1)$ is defined in (21). To show that such a fixed point exists, we need the next lemma which we prove later for uninterrupted flow of argument.

Lemma 9 *Let $\psi : (\frac{\ell}{1-\delta}, \frac{1}{1-\delta}) \to \mathcal{C}_{[0,1]}$ be a mapping such that $\psi(v_1) = V^{\dagger}_{v_1}$ where $\mathcal{C}_{[0,1]}$ is the set of all continuous functions on $[0, 1]$. Then, ψ is continuous in v_1 under the sup norm at any $v_1 > \frac{\ell}{1-\delta}$.*

Note that, as $v_1 \to \frac{\ell}{1-\delta}$, $\mu^{\dagger}_1(v_1)$ converges to a limit strictly greater than 0. Since the right derivative of $p_G(\mu, y^{\dagger}_{v_1}(\mu))$ with respect to μ is uniformly bounded away from 0 at $\mu = 0$, so is the right derivative of $V^{\dagger}_{v_1}(\mu)$. Since $V^{\dagger}_{v_1}(0) > \frac{\ell}{1-\delta}$ by (16) and $V^{\dagger}_{v_1}$ is strictly increasing, therefore, $V^{\dagger}_{v_1}(\mu^{\dagger}_1(v_1)) > v_1$ for v_1 sufficiently close to $\frac{\ell}{1-\delta}$. On the other hand, as $v_1 \to \frac{1}{1-\delta}$, since $\mu^i < 1$ we have $V^{\dagger}_{v_1}(\mu^{\dagger}_1(v_1)) \leq V^{\dagger}_{v_1}(\mu^i) < V^{\dagger}_{v_1}(1) \leq v_1$ for v_1 sufficiently close to $\frac{1}{1-\delta}$ by (16). Then, since $\mu^{\dagger}_1(v_1)$ is continuous in v_1 from (21) and ψ is continuous by Lemma 9, we must have $V^{\dagger}_{v_1}(\mu^{\dagger}_1(v_1)) = v_1$ for at least one $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$.

Let μ_1 and v_1 denote a pair of stationary default reputation level and value, i.e., $v_1 = V^{\dagger}_{v_1}(\mu_1)$ and $\mu_1 = \mu^{\dagger}_1(v_1)$. Note that to establish a stationary equilibrium, we still need to show that it is optimal for h-sellers to always report truthfully as long as $\mu \geq \mu_1$. Since the continuation value of h-seller after cheating is the equilibrium value of the default level μ_1, $V^{\dagger}_h(\mu_1)$, rather than $V^{\dagger}_h(0)$, the optimality condition of h-seller is more difficult to verify than when restarting is impossible. In fact, it has not been proved that for all stationary pairs of μ_1 and v_1, truthful announcement for all $\mu \geq \mu_1$ is optimal for h-sellers when ℓ-sellers announce according to $y^{\dagger}_{v_1}(\mu)$ for $\mu \geq \mu_1$.

However, the proof of [S] included in the proof of Theorem 1 relies on $V^*_\ell(0)$ being a constant, rather than $V^*_\ell(0) = \frac{\ell}{1-\delta}$ and consequently, applies analogously to $V^{\dagger}_h(\mu)$ defined as per (45) with y^* replaced by $y^{\dagger}_{v_1}$ for $\mu > \mu_1$. As a result, if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$, it constitutes an equilibrium for ℓ-sellers to announce according to $y^{\dagger}_{v_1}(\mu)$ and h-sellers honestly for $\mu \geq \mu_1$ for any stationary pair μ_1 and v_1, provided that $\delta < 1$ is sufficiently large so that, in particular, $\delta(V_h^\dagger(1)-V_h^\dagger(\mu_1)) \geq 1$.[25] It may be worth mentioning that this is a sufficient condition, so stationary equilibria in which h-sellers behave honestly may exist in a wider class of environments.

Finally we prove Lemma 9. Since continuity under the sup norm requires uniform convergence, the possibility of a fixed point having unbounded derivative poses a potential problem. The bulk of the proof evolves around how to circumvent this problem. We start with two preliminary lemmas asserting that $p_G^*(\mu)$ is of bounded variation on $[\varepsilon, 1]$ for any $\varepsilon > 0$ (Lemma A1) and consequently, so is the fixed point $V_{v_1}^\dagger$ (Lemma A2).

Lemma A1 *For any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\forall v_1 \in [\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, $\forall V \in \mathcal{F}_{v_1} \cap \mathcal{C}_{[0,1]}$, $\forall \mu$ and $\mu' \in (\varepsilon, \bar{\mu}^\dagger)$,*

$$\frac{p_G(\mu', y_V^\dagger(\mu')) - p_G(\mu, y_V^\dagger(\mu))}{\mu' - \mu} \leq M_\varepsilon. \tag{63}$$

Proof. Note from (1) that we can find $k > 0$ such that $\frac{\partial p_G}{\partial \mu} > 0$ is bounded above uniformly by k, and $\frac{\partial p_G}{\partial y} < 0$ is bounded below uniformly by $-k$. Suppose $\mu < \mu'$ without loss of generality. If $y_V^\dagger(\mu') \geq y_V^\dagger(\mu)$, then $\frac{p_G(\mu', y_V^\dagger(\mu')) - p_G(\mu, y_V^\dagger(\mu))}{\mu'-\mu} < k$ because p_G decreases in y, proving (63).

Now suppose that $y_V^\dagger(\mu') < y_V^\dagger(\mu)$. Note that one can find $k_\varepsilon,\ \tilde{k}_\varepsilon > 0$ such that

$$\begin{aligned}
\frac{\partial \pi_{Bb}(\mu, y)}{\partial \mu} &= \frac{(1-h)(1-\ell)(1-y)}{[\mu(1-h) + (1-\mu)(1-\ell)(1-y)]^2} < k_\varepsilon \\
\frac{\partial \pi_{Bb}(\mu, y)}{\partial y} &= \frac{(1-h)(1-\ell)(1-\mu)\mu}{[\mu(1-h) + (1-\mu)(1-\ell)(1-y)]^2} > \tilde{k}_\varepsilon
\end{aligned}$$

for all $\mu > \varepsilon$ and $y \in [0,1]$. Thus, recalling that $\pi_{Bb}\big(\mu, y_V^\dagger(\mu)\big)$ is nondecreasing, we deduce that

$$0 \leq \pi_{Bb}(\mu', y_V^\dagger(\mu')) - \pi_{Bb}(\mu, y_V^\dagger(\mu)) < k_\varepsilon(\mu' - \mu) + \tilde{k}_\varepsilon\big(y_V^\dagger(\mu') - y_V^\dagger(\mu)\big),$$

using the facts that $y_V^\dagger(\mu') < y_V^\dagger(\mu)$ and $\mu < \mu'$, and consequently,

$$y_V^\dagger(\mu') - y_V^\dagger(\mu) > -\frac{k_\varepsilon}{\tilde{k}_\varepsilon}(\mu' - \mu).$$

Therefore, we have

$$\begin{aligned}
p_G(\mu', y_V^\dagger(\mu')) - p_G(\mu, y_V^\dagger(\mu)) &< k(\mu' - \mu) - k\big(y_V^\dagger(\mu') - y_V^\dagger(\mu)\big) \\
&< k\Big(1 + \frac{k_\varepsilon}{\tilde{k}_\varepsilon}\Big)(\mu' - \mu).
\end{aligned}$$

We complete the proof by setting $M_\varepsilon = k\big(1 + \frac{k_\varepsilon}{\tilde{k}_\varepsilon}\big)$. ∎

[25] The proof is omitted because it is the same as the proof of [S] with obvious changes, such as v_1 and $\bar{\mu}^\dagger$ in place of $V_\ell^*(0)$ and $\bar{\mu}$, respectively.

Lemma A2 *For any $\varepsilon > 0$ and $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$,*

$$D^+V^\dagger_{v_1}(\mu) := \limsup_{\mu' \downarrow \mu} \frac{V^\dagger_{v_1}(\mu') - V^\dagger_{v_1}(\mu)}{\mu' - \mu} \;\leq\; \frac{M_\varepsilon}{1-\delta h} \quad \textit{if} \;\; \mu > \varepsilon, \tag{64}$$

where $V^\dagger_{v_1}$ is the fixed point of the operator T_{v_1}.

Proof. For given v_1 there exists $\underline{\mu} > 0$ defined by $V^\dagger_{v_1}(\pi^*_{Gg}(\underline{\mu})) = v_1$, so that

$$V^\dagger_{v_1}(\mu) = \begin{cases} p_G(\mu, y^\dagger_{v_1}(\mu)) + \delta v_1 \quad \text{if} \quad \mu \leq \underline{\mu} \\ p_G(\mu, y^\dagger_{v_1}(\mu)) + \delta\left(\ell V^\dagger_{v_1}(\pi^*_{Gg}(\mu)) + (1-\ell)v_1\right) \quad \text{if} \quad \mu \geq \underline{\mu}. \end{cases} \tag{65}$$

To reach a contradiction, suppose that for any $K > 0$ one can find $\mu_1 > \varepsilon$ such that $D^+V^\dagger_{v_1}(\mu_1) > K$. Then, since $\pi^*_{Gg}(\mu)$ is differentiable and $\frac{\ell}{h} \leq \frac{\partial \pi^*_{Gg}(\mu)}{\partial \mu} \leq \frac{h}{\ell}$, (63) and (65) would imply that $\mu_1 > \underline{\mu}$ when K is sufficiently large and that one can construct a sequence $\mu_n \to 1$ where $\mu_n = \pi^*_{Gg}(\mu_{n-1})$. Since there is $\tau < \infty$ such that $\pi^\tau_{Gg}(\mu) > \bar{\mu}^\dagger$ for any $\mu > \varepsilon$, by choosing K arbitrarily large, one can ensure that $D^+V^\dagger_{v_1}(\pi^\tau_{Gg}(\mu))$ is arbitrarily large. But, this is impossible because $D^+V^\dagger_{v_1}(\mu)$ is bounded for $\mu > \bar{\mu}^\dagger$ as can be verified from

$$V^\dagger_{v_1}(\mu) = \Big[\sum_{t=0}^{\infty} \ell^t \delta^t p_G(\pi^t_{Gg}(\mu), 1)\Big] + \delta v_1(1-\ell)\sum_{t=0}^{\infty} \ell^t \delta^t, \tag{66}$$

a formula adapted from (**??**) for $V^\dagger_{v_1}(\mu)$ for $\mu > \bar{\mu}^\dagger$. Hence, we conclude that $D^+V^\dagger_{v_1}(\mu)$ is uniformly bounded for $\mu > \varepsilon$ and thus, (63) and (65) imply

$$\begin{aligned} D^+V^\dagger_{v_1}(\mu) &\leq M_\varepsilon + \ell\delta\Big(\sup_{\mu>\varepsilon} D^+V^\dagger_{v_1}(\mu)\Big)\Big(\max_{\mu} \frac{\partial \pi^*_{Gg}(\mu)}{\partial \mu}\Big) \\ &\leq M_\varepsilon + h\delta\Big(\sup_{\mu>\varepsilon} D^+V^\dagger_{v_1}(\mu)\Big) \end{aligned}$$

for $\mu > \varepsilon$. Thus, $D^+V^\dagger_{v_1}(\mu) \leq \frac{M_\varepsilon}{1-\delta h}$ if $\mu > \varepsilon$. ∎

Next, choose $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$. Notice that for a sufficiently small $\eta > 0$, in particular smaller than $v_1 - \frac{\ell}{1-\delta}$, the operator T_{v_1} can be extended to $\mathcal{F}^\eta_{v_1} \cap \mathcal{C}_{[0,1]}$ where $\mathcal{F}^\eta_{v_1} := \cup_{v_1-\eta \leq v \leq v_1+\eta} \mathcal{F}_v$. As an intermediate step, we need

Lemma A3 *For $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, the operator*

$$T_{v_1} : \mathcal{F}^\eta_{v_1} \cap \mathcal{C}_{[0,1]} \to \mathcal{C}_{[0,1]} \textit{ is continuous in sup norm.} \tag{67}$$

Proof. Consider $V, V' \in \mathcal{F}^\eta_{v_1} \cap \mathcal{C}_{[0,1]}$ such that $\max_{\mu \in [0,1]} |V'(\mu) - V(\mu)| < \epsilon$. Since $y^\dagger_V(\mu)$ and $y^\dagger_{V'}(\mu)$ are, by construction, the solutions to

$$\min_{0 \leq y \leq 1} \left| p_G(\mu, y) - \delta\big(\max\{v_1, V(\pi_{Bb}(\mu, y))\} - v_1\big)\right| \tag{68}$$

and the same equation with V' instead of V, respectively, it follows that $|p_G(\mu, y^\dagger_{V'}(\mu)) - p_G(\mu, y^\dagger_V(\mu))| < \epsilon$. From (56), therefore, we deduce that

$$\max_{\mu \in [0,1]} |T_{v_1}(V')(\mu) - T_{v_1}(V)(\mu)| < \epsilon + \delta\epsilon,$$

which establishes (67). ∎

Given $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$ and η small as specified above, consider small $|\kappa| < \eta/2$ and any $V \in \mathcal{F}^{\eta}_{v_1} \cap \mathcal{F}^{\eta}_{v_1+\kappa} \cap \mathcal{C}_{[0,1]}$. By (68), the value of $p_G(\mu, y^{\dagger}_V(\mu))$ differs when calculated for T_{v_1} and when calculated for $T_{v_1+\kappa}$, and the difference is at most $\delta\kappa$. Thus, from (56),

$$T_{v_1}(V)(\mu) - 2|\delta\kappa| \leq T_{v_1+\kappa}(V)(\mu) \leq T_{v_1}(V)(\mu) + 2|\delta\kappa| \quad \forall \mu \in [0,1]. \tag{69}$$

In particular, observe that

$$T_{v_1}(V^{\dagger}_{v_1+\kappa})(\mu) - 2|\delta\kappa| \leq T_{v_1+\kappa}(V^{\dagger}_{v_1+\kappa})(\mu) = V^{\dagger}_{v_1+\kappa}(\mu) \leq T_{v_1}(V^{\dagger}_{v_1+\kappa})(\mu) + 2|\delta\kappa|.$$

Finally, to prove continuity of ψ at v_1, we decompose the argument into two parts: First, we prove uniform convergence of functions $\psi(v_1 + \kappa) = V^{\dagger}_{v_1+\kappa}$ to $\psi(v_1) = V^{\dagger}_{v_1}$ as $\kappa \to 0$ on intervals $[\varepsilon, 1]$, then do the same separately on $[0, 2\varepsilon]$. The continuity will be established by combining the two parts.

We know from Lemma A2 that on the interval $[\varepsilon, 1]$, the function $V^{\dagger}_{v_1+\kappa}$ is K_{ε}-Liptchitz where $K_{\varepsilon} = \frac{M_{\varepsilon}}{1-\delta h}$. Then from Ascoli-Arzelà Theorem (see Royden (1988)), the subset consisting of all K_{ε}-Lipschitz function of $\mathcal{F}^{\eta}_{v_1}$ is compact under the sup norm. Hence, there exists a sequence of fixed points $V^{\dagger}_{v_1+\kappa}$ such that, when restricted to the domain $[\varepsilon, 1]$, it converges as $\kappa \to 0$ to a limit, denoted by $W^{[\varepsilon,1]}_{v_1}$, where $W^{[\varepsilon,1]}_{v_1}$ is continuous on $[\varepsilon, 1]$ and

$$V^{\dagger}_{v_1+\kappa} \xrightarrow{unif} W^{[\varepsilon,1]}_{v_1} \text{ under the sup norm on } [\varepsilon, 1] \text{ for any } \varepsilon > 0. \tag{70}$$

Let $V^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ denote $V^{\dagger}_{v_1+\kappa}$ restricted on $[\varepsilon, 1]$ and let $\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ denote the continuous linear extension of $V^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ on $[0, \varepsilon]$. Then, by (67) and (69),

$$T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu) \leq \lim_{\kappa\to 0} T_{v_1+\kappa}(\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu) \leq T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu). \tag{71}$$

Note that $T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu)$ for each μ is fully determined by $\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ restricted on $[\mu, 1]$ according to (56), and the same is true for $T_{v_1+\kappa}(\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})$. Since $\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa} = V^{\dagger}_{v_1+\kappa}$ on $[\varepsilon, 1]$ by definition, therefore, (70) and (71) imply that

$$T_{v_1}(\widetilde{W}^{[\varepsilon,1]}_{v_1})(\mu) \leq \widetilde{W}^{[\varepsilon,1]}_{v_1}(\mu) \leq T_{v_1}(\widetilde{W}^{[\varepsilon,1]}_{v_1})(\mu) \quad \text{for all} \quad \mu \in [\varepsilon, 1],$$

where $\widetilde{W}^{[\varepsilon,1]}_{v_1}$ is the continuous linear extension of $W^{[\varepsilon,1]}_{v_1}$ on $[0, \varepsilon]$. Since $\varepsilon > 0$ is arbitrary and $V^{\dagger}_{v_1}$ is the only function V that satisfies $T_{v_1}(V)(\mu) = V(\mu)$ on $[\varepsilon, 1]$ for all $\varepsilon \in (0, 1)$ by uniqueness of the fixed point of T_{v_1}, it further follows that $\widetilde{W}^{[\varepsilon,1]}_{v_1} = V^{\dagger}_{v_1}$ on $[\varepsilon, 1]$, i.e., $W^{[\varepsilon,1]}_{v_1}$ coincides with $V^{\dagger}_{v_1}$ on $[\varepsilon, 1]$. From (70), therefore,

$$V^{\dagger}_{v_1+\kappa} \xrightarrow{unif} V^{\dagger}_{v_1} \text{ under the sup norm on } [\varepsilon, 1] \text{ for any } \varepsilon > 0. \tag{72}$$

Note, however, that this is not sufficient for uniform convergence on $[0, 1]$. Hence, choose $\breve{\mu} > 0$ such that $V^{\dagger}_{v_1}(\pi^*_{Gg}(\breve{\mu})) < v_1$. Then, because $V^{\dagger}_{v_1+\kappa}$ converges to $V^{\dagger}_{v_1}$ under the sup norm on $[\frac{\breve{\mu}}{2}, 1]$ by (72), we have $V^{\dagger}_{v_1+\kappa}(\pi^*_{Gg}(\breve{\mu})) < v_1 + \kappa$ for sufficiently small

κ. But this implies that $V^{\dagger}_{v_1+\kappa}(\pi^*_{Gg}(\mu)) < v_1 + \kappa$ for all $\mu \leq \breve{\mu}$ for sufficiently small κ, and consequently, $V^{\dagger}_{v_1+\kappa}(\mu) = p_G(\mu, 1) + \delta(v_1 + \kappa)$ on $[0, \breve{\mu}]$, which converges uniformly to $V^{\dagger}_{v_1}(\mu) = p_G(\mu, 1) + \delta v_1$. Thus, $V^{\dagger}_{v_1+\kappa} \xrightarrow{unif} V^{\dagger}_{v_1}$ under the sup norm on $[0, \breve{\mu}]$. Combining this with (72) when $\varepsilon = \breve{\mu}/2$, we obtain uniform convergence under the sup norm on the entire domain $[0, 1]$, which proves continuity of ψ at v_1 and thus, Lemma 9. This completes the proof of Theorem 3. ■

References

Atkeson, A., C. Hellwig, and G. Ordonez (2010), “Optimal regulation in the presence of reputation concerns,” mimeo, UCLA and Yale Univ.

Bajari, P., and A. Hortacsu, (2004), “Economic Insights from Internet Auctions,” *Journal of Economic Literature*, **42**, 457–486.

Bar-Isaac, H. (2003), “Reputation and Survival: learning in a dynamic signalling model”, *Review of Economic Studies*, **70**, 231–251.

Bar-Isaac H. and S. Tadelis, (2008), ”Seller Reputation”, *Foundations and Trends in Microeconomics*, **4**, 273–351

Benabou, R. and G. Laroque (1992), “Using Privileged Information to Manipulate Markets: Insiders, Gurus and Credibility,” *Quarterly Journal of Economics,* **107**, 921–958.

Billingsley, P. (1999), *Convergence of Probability Measures,* 2nd ed., John Wiley & Sons, Inc.

Cabral, L., and A. Hortacsu, (2008), “The Dynamics of Seller Reputation: Theory and Evidence from eBay,” mimeo, New York University.

Canals-Cerda, J. (2008), “The Value of a Good Reputation Online: An Application to Art Auctions,” mimeo, Federal Reserve Bank of Philadelphia

Cripps, M., Mailath, G., and L. Samuelson (2004), “Imperfect Monitoring and Impermanent Reputations,” *Econometrica,* **72**, 407–432.

Dellarocas, C. (2003). “The Digitization of Word of Mouth: Promise and Challenges of Online Feedback Mechanisms,” *Management Science*, **49**, 1407–1424.

Dellarocas, C. (2006), ”Reputation Mechanisms”, *Handbook on Economics and Information Systems* (T. Hendershott, ed.), Elsevier Publishing.

Diamond, D. (1989), ”Reputation Acquisition in Debt Market”, *J. of Polit. Econ.*, **97**, 828-862.

Ely, J., and J. Valimaki (2003), “Bad Reputation,” *Quar. J. of Econ.,* **118**, 785–814.

Fan, K. (1952), “Fixed point and minimax theorems in locally convex topological linear spaces,” *Proc. Nat. Acad. Sci. U.S.A.,* **38**, 121–126.

Friedman E., and P. Resnick (2001), “The Social Cost of Cheap Pseudonyms,” *Journal of Economics and Management Science,* **10**, 173–199

Fudenberg, D., and D. Levine (1989), “Reputation and equilibrium selection in games with a patient player,” *Econometrica,* **57**, 759–778.

Glicksberg, I. L. (1952), “A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points,” *Proc. Amer. Math. Soc.,* **3**, 170–174.

Houser, D. and J. Wooders (2006) “Reputation in Auctions: Theory, and Evidence from eBay”, *Journal of Economics and Management Strategy,* **15**, 353-369.

Jin, G. and A. Kato (2006), "Price, Quality and Reputation: Evidence from an Online Field Experiment," *RAND Journal of Economics,* **37**, 983–1005.

Jullien, B. and I.-U. Park (2009), "Seller Reputation and Trust in Pre-Trade Communication," IDEI Working Papers 564, Toulouse School of Economics.

Kalyanam, K. and S. McIntyre (2001), "Returns to Reputation in Online Auction Markets," mimeo, Santa Clara U.

Kreps, D. and R. Wilson (1982), "Reputation and Imperfect Information," *Journal of Economic Theory,* **27**, 245–252.

Livingston, J. (2005), "How Valuable is a Good Reputation? A Sample Selection Model of Internet Auctions," *The Review of Economics and Statistics,* **87**, 453-465.

Mailath, G. and L. Samuelson (2001), "Who Wants a Good Reputation," *Review of Economic Studies,* **68**, 415–441.

Mailath, G. and L. Samuelson (2006), *Repeated Games and Reputations: Long-Run Relationships,* Oxford University Press.

Mathis,J., McAndrews, J. and J.C. Rochet (2009), "Rating the raters: are reputation concerns powerful enough to discipline rating agencies?," *J. of Monetary Economics,* **56**, 657–674.

McLennan, A. and I.-U. Park (2007), "The Market for Liars: Reputation and Auditor Honesty," mimeo, University of Bristol.

Melnik, M. and J. Alm (2002) "Does a Seller's eCommerce Reputation Matter? Evidence from eBay Auctions," *Journal of Industrial Economics*, **50**, 337-349.

Milgrom, P. and D.J. Roberts (1982), "Predation, Reputation and Entry Deterrence," *Journal of Economic Theory,* **27**, 280–312.

Morris, S. (2001), "Political Correctness," *Journal of Political Economy,* **109**, 231–265.

Ottaviani, M. and P. Sorensen (2001) "Information aggregation in debate: who should speak first", *Journal of Public Economics*, **81**, 393-42.

Ottaviani, M. and P. Sorensen (2006), "Reputational cheap talk," *RAND Journal of Economics,* **37**, 155–175.

Park, I.-U. (2005), "Cheap-talk referrals of differentiated experts in repeated relationships," *RAND Journal of Economics,* **36**, 391–441.

Resnick, P, R. Zeckhauser, J. Swanson, and K. Lockwood (2006), "The Value of Reputation on Ebay: A Controlled Experiment," *Experimental Economics*, **9**, 79-101.

Royden H.L. (1988), *Real Analysis,* 3rd ed., Macmillan Publishing Company, New York.

Sobel, J. (1985), "A Theory of Credibility," *Review of Economic Studies,* **52**, 557-573.

Tadelis, S. (1999), "What's in a Name? Reputation as a Tradeable Asset," *American Economic Review,* **89**, 548–563.