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Pubertal Growth of the Cephalometric Point Gnathion: Multilevel Models for Boys and Girls

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ABSTRACT Two-level polynomial models are used to summarize the amount Sella-Gnathion (S-Gn) and direction Nasion-Sella-Gnathion (N-S-Gn) of growth changes for the cephalometric landmark gnathion. Growth descriptions pertain to a mixed longitudinal sample of 209 French-Canadian children 10–15 years of age. The boy's growth curve attains mean minimum prepubertal velocity at 10.8 years and maximum pubertal velocity at 14.1 years. The girl's curve follows a cubic pattern, attaining maximum pubertal velocity at 12.1 years. Boys are larger than girls throughout the age range. Variation between-subjects increases with age in a curvilinear fashion. Growth direction of gnathion is more horizontally directed for girls than boys. Small but significant changes in growth direction occur between 10 and 15 years of age.

While much progress has been made in the analysis of somatic growth (Roche, 1986), the application of mathematical models to serial craniofacial data remains rudimentary. Polynomial models have been used to investigate sexual dimorphism in mandibular growth during childhood (Buschang et al., 1986a) and to compare the craniofacial growth of adolescent males with normal and class II malocclusion (Buschang et al., 1986b). These studies demonstrate that polynomial regression models are particularly appropriate for studying craniofacial growth; they make fewer requirements and are more flexible than nonlinear parametric and nonparametric models, and they are able to incorporate other explanatory variables (Goldstein,1986a).

A new class of multilevel models have been introduced (Strenio et al., 1983; Goldstein, 1986b) which further refine the analysis of serial data. These models extend to *unadjusted mixed longitudinal data*, making it possible to efficiently estimate growth parameters of individuals with missing observations or unadjusted series of measurements. They provide descriptions of average growth tendencies and their between-subject variabilities.

Two-level polynomial models were recently used to study the mandibular growth between 6 and 15 years of French-Canadian girls (Buschang et al., 1988). Peak childhood and adolescent velocities were estimated at 7.5 and 12.7 years, respectively. Due to sampling limitations some of the model's random coefficients remained fixed, resulting in biased estimations of the *individuals'* growth spurts.

The present study was undertaken to more precisely estimate the timing and variability of mandibular growth changes at gnathion during adolescence. The point gnathion was chosen because 1) it is one of the most reliable cephalometric landmarks available (Buschang et al., 1987a), and 2) it is commonly used for clinical evaluation of mandibular position (Downs, 1948; Enlow, 1982; Graber, 1972; Salzmann, 1966). In lieu of substantially increasing sample sizes to estimate the necessary parameters, multilevel models were fitted to mandibular growth data of adolescent children 10 to 15 years of age.

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MATERIALS AND METHODS

The data are derived from serial lateral cephalograms collected by the Human Growth Research Center, Université de Montréal. They pertain to French-Canadian boys and girls drawn from three school districts in Montreal chosen to represent the different socioeconomic sectors of the larger population (Demirjian et al., 1971). Data for a mixed longitudinal sample of 105 girls (n = 579) and 104 boys (n = 597) followed between 10 and 15 years of age were available for the analyses. Of the 209 children in the study, 67% had complete serial data, 29% had one missing observation, and 4% had two missing observations.

The cephalograms were traced and digitized by the same technician (L.L.). The analyses pertain to the cephalometric landmark gnathion defined as the point on the symphysis formed by bisecting the projections of the mandibular plane and a plane perpendicular to the mandibular plane and tangential to the most anterior point on the mandible. Technical reliability (Buschang et al., 1987a) has been estimated at 98.4 and 98.8% for the horizontal and vertical aspects of gnathion, respectively. The growth of gnathion was evaluated relative to the cephalometric point sella; the sella-nasion (S-N) reference plane was used for orientation. All measures were corrected for radiographic enlargement (0.8892%). To better distinguish the *amount* and *direction* of mandibular growth from S-N, the digitized, rectangular (X,Y) coordinates were transformed into polar $((\gamma,\theta))$ form. As such, the distance sella-gnathion (S-Gn), hereafter referred to as gnathion length, describes the amount of growth and the angle nasion-sella-gnathion (N-S-Gn) describes the direction of growth.

Previous analyses (Buschang et al., 1987b, 1988) indicate that the mean preadolescent minimum growth velocity for gnathion occurs at approximately 10.1 years for girls and 11.7 years for boys. To allow for variation in the timing of this event and for the adolescent spurt, we might expect to fit at least a cubic polynomial for girls and a quartic polynomial for boys. On this basis, the following polynomial function was fitted to the individuals' serial data:

$$Y_{it} = \sum_j \beta_{ij} X_t^j + \epsilon_{it},\ t = 1 \ldots n_i$$

where, Y_{ij} is the observed gnathion length or growth direction for the i^{th} child on the t^{th} measurement occasion, j indexes the polynomial coefficients, X is the age at occasion t, and it is a random term describing the i^{th} child's residual at the t^{th} occasion.

The model partitions variation into two levels, with subjects at the higher level and measurement occasions at the lower level (see Appendix A). The β_{ij} are assumed to vary randomly over individuals. Iterative generalized least squares (Goldstein, 1986b, 1987) was used to estimate the model's parameters.

RESULTS

Table 1 describes the mean growth curves and variation for S-Gn. The boys' curve follows a quartic polynomial in age, which indicates that growth velocity changes direction three times between 10 and 15 years of age (Fig. 1). When the distribution of ages of zero acceleration are estimated (see appendix) we can see that minimum prepubertal velocity is attained at a mean age of 10.8 years and maximum adolescent velocity at 14.1 years.

The girl's cubic polynomial shows that growth accelerates until peak adolescent velocity, at the mean age of 12.1 years, and then decelerates. Peak adolescent velocity is approximately 1.6 times greater for boys than girls. Due to the different maturity patterns, gnathion length remains approximately 4 mm larger for boys than girls between 10 and 12.5 years (Fig. 2). Size differences increase thereafter, as growth in gnathion length continues to accelerate for boys and decelerates for girls. Boys are approximately 8 mm larger than girls at 15 years of age.

We have been able to fit only a single parameter (σ_0^2) to describe within-subject variation. Variation between subjects changes with age in a curvilinear manner (Fig. 3). The quartic and cubic coefficients remained fixed for boys and girls, respectively; the other terms vary randomly across subjects. As expected, the estimated standard deviations closely follow the observed growth velocities. For boys, correlations between the estimated parameters range from moderately low (−0.38) to moderately high (−0.81). Girls show a moderate correlation (−0.53) between their intercept and quadratic terms and a moderately low association (0.35) between their linear and quadratic terms.

Age changes in growth direction are small and relatively simple (Table 2). For boys, the angle (N-S-Gn) closes slightly, but significantly, between 10 and 12 years of age and then opens up or rotates positively during the adolescent spurt (Fig. 4). Growth direc-

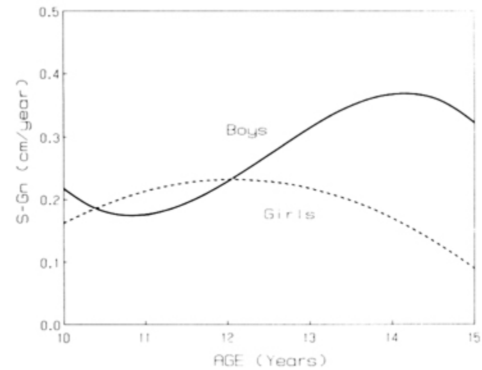

Fig. 1. Velocity curve for sella-gnathion (S-Gn) from 10 to 15 years of age.

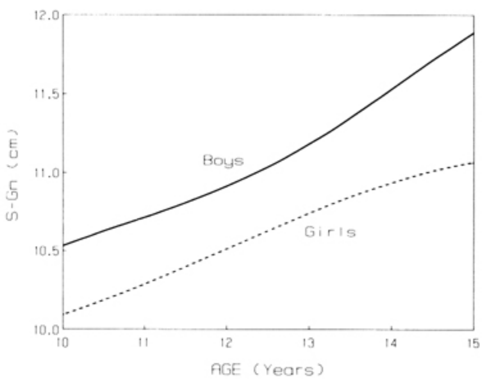

Fig. 2. Distance curve for sella-gnathion (S-Gn) from 10 to 15 years of age.

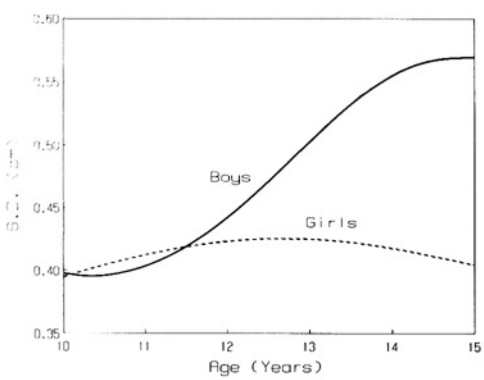

Fig. 3. Standard deviations from 10 to 15 years of age for sella-gnathion (S-Gn).

tion of gnathion for girls is more vertically oriented than for boys; the angle increases regularly (linearly) with age. Variation in growth direction is pronounced. The between-subject standard deviations range from 3.0° at 10 years to 3.7° at 15 years (Fig. 5). Increases in variance are especially marked for girls, which might be attributed to the positive correlations found between growth direction at 13 years (intercept) and velocity, which are moderately low for girls (0.47) and not significant for boys. The associations indicate that the direction of mandibular growth for horizontal and vertical grower tends to diverge with increasing age.

DISCUSSION

The findings provide detailed descriptions of adolescent growth changes for the cephalometric landmark gnathion. Morphologically, changes in the position of gnathion can be explained, primarily, by mandibular displacement and, secondarily, by remodelling in the symphyseal region (Björk, 1969; Enlow, 1982). As such, gnathion length (S-Gn) serves a summary variable incorporating the growth, displacement, and remodelling changes occurring between the points sella and gnathion. Importantly, sella-gnathion, otherwise referred to as the Y-axis (Downs, 1948), is well accepted and commonly used by orthodontists to ascertain the position of the chin, the direction of mandibular growth, and retraction or protraction of the mandible (Graber, 1972; Salzmann, 1966). As such, the results presented herein provide highly accurate and flexible descriptions that can be easily applied to evaluate the movement of gnathion relative to the anterior cranial base, which is well established as one of the most stable regions during adolescence due to its endochondral dominance and early completion.

The growth curve for gnathion follows the general pattern established for somatic measures (Tanner, 1962). Individual children, however, display considerable variation in the timing and amount of growth changes at gnathion. The multilevel procedure provides "posterior" estimates of the polynomial coefficients for each individual based on the estimated variance/covariance matrix of the random coefficients. Such estimates are available even when only a limited number of data points have been obtained for an individual (Goldstein, 1987). Fig. 7 shows velocity curves for gnathion length derived from the polynomial coefficient of four boys who were chosen to demonstrate the normal vari-

TABLE 1. Pubertal growth of sella-gnathion (cm) for French-Canadians 10–15 years of age

Explanatory variables	Boys		Girls	
	Estimates	Standard errors	Estimates	Standard errors
Fixed coefficients				
Intercept	11.183		10.739	
Age	0.3151	0.011796	0.21737	0.0060759
Age^2	0.040337	0.0052477	−0.015519	0.0025509
Age^3	−0.0055298	0.0020993	−0.005487	0.00108
Age^4	−0.0027282	0.00079824		
Random Coefficients				
σ_0^2	0.0050169	0.00051295	0.0067038	0.00056838
$\sigma^2_{\mu,0}$	0.24791	0.034716	0.17415	0.024397
$\sigma^2_{\mu,1}$	0.0099013	0.0016677	0.001174	0.00027215
$\sigma^2_{\mu,2}$	0.00071304	0.00015022	0.00013799	0.000052262
$\sigma^2_{\mu,3}$	0.00010528	0.000027841		
$\sigma_{\mu,01}$	0.031829	0.0062369	−0.0011387	0.0018361
$\sigma_{\mu,02}$	−0.0050939	0.0016982	−0.0026208	0.00084279
$\sigma_{\mu,03}$	−0.0025762	0.00073368		
$\sigma_{\mu,12}$	−0.0012015	0.0003845	0.00014231	0.000090863
$\sigma_{\mu,13}$	−0.00083062	0.00019255		
$\sigma_{\mu,23}$	0.00021666	0.000057105		
$\rho_{\mu,01}$	0.642		−0.080	
$\rho_{\mu,02}$	−0.383		−0.534	
$\rho_{\mu,03}$	−0.504			
$\rho_{\mu,12}$	−0.452		0.354	
$\rho_{\mu,13}$	−0.813			
$\rho_{\mu,23}$	0.791			
Iterations	5		6	
Measurements	597		579	
Subjects	104		105	
Age measured from	13 years		13 years	
Relative accuracy for convergence	10^{-3}		10^{-3}	

TABLE 2. Pubertal changes in nasion-sella-gnathion (degrees) for French-Canadians 10–15 years of age

Explanatory variables	Boys		Girls	
	Estimates	Standard errors	Estimates	Standard errors
Fixed coefficients				
Intercept	67.762		68.8876	
Age	0.10211	0.03137	0.13429	0.038479
Age^2	0.03879	0.0093931		
Random coefficients				
σ_0^2	0.0050169	0.00051295	0.0067038	0.00056838
$\sigma^2_{\mu,0}$	10.839	1.5115	10.873	1.5115
$\sigma^2_{\mu,1}$	0.073163	0.012968	0.12935	0.021501
$\sigma_{\mu,01}$	0.19412	0.10077	0.56186	0.13926
$\sigma_{\mu,01}$	0.218		0.474	
Iterations		4		6
Measurements		597		579
Subjects		104		105
Age measured from		13 years		13 years
Relative accuracy for convergence		10^{-3}		10^{-3}

ation in pubertal growth. Growth differences are clearly reflected in their estimated coefficients. For example, the standardized cubic and quartic terms for subjects 8 (1.7 and 1.8) and 60 (−3.2 and −2.2) indicate that their pubertal peaks occur later and earlier than average, respectively. The standardized coefficients for subjects 58 and 79, whose ages of peak velocity approximate the mean age reported for the sample, are within 0.9 standard deviation units from the mean.

The results also show that the direction of growth for gnathion changes regularly for girls and is curvilinear for boys. A larger sample of girls may also reveal more complex changes in growth direction. The point gnathion undergoes a small but significant posterior rotation during adolescence, which may be partially attributed to remodelling changes at nasion (Björk and Skieller, 1976). It should be emphasized that gnathion posterior rotation when evaluated relative to stable references structure in the anterior cranial base and vault (Buschang et al., 1987b) shows the same pattern.

Pubertal spurts occurred for all of the children studied. As previously reported, the spurt is more intense and occurs approximately two years later for boys than for girls (Nanda, 1955; Bambha, 1961; Hunter, 1966; Tracy and Savara, 1966; Savara and Tracy, 1967; Lewis et al., 1982). Pubertal spurts occur somewhat later for French-Canadian children than for other samples (Table 3). Differences may be attributed to the various measurements reported and to the criteria, often subjective, previously used to identify growth spurts.

It is also of interest to compare the results with somatic growth, for which a variety of mathematical models have been fitted (Roche, 1986). Our estimates for the age of prepubertal minimum velocity for boys and for pubertal maximum velocity for boys and girls are only slightly later than values reported in the more recent studies for growth in stature (Tanner et al., 1976; Largo et al., 1978; Preece and Baines, 1978; Preece and Heinrich, 1981; Gasser et al., 1984). Estimates of peak adolescent velocity for growth in sitting height and biacromial diameter (Tanner et al., 1976) are in very close agreement with our estimates for gnathion length, suggesting that the timing of mandibular growth more closely follows the protracted pattern of the trunk. It has been previously shown that the maturity status of the mandible lags behind the maturity status of stature during adolescence (Buschang et al., 1983).

TABLE 3. Timing (years) of maximum pubertal growth

		Boys			Girls		
References	Measure	N	Mean age	S.D.	N	Mean age	S.D.
Mandibular growth							
Fukuhara & Matsumoto ('68)[1]	S-Gn	24	12.5		27	11.5	
Thompson et al. ('76)	Cn-Gn				111	11.4	1.3
Lewis et al. ('82)[2]	Ar-Gn	29	13.1	1.4	20	11.6	1.6
	Ar-Go	25	13.5	1.7	18	11.8	1.3
	Go-Gn	30	13.2	1.4	22	11.3	1.8
Brown et al. ('71)	N-Gn	27	13.8	1.4	16	12.2	1.3
	Ar-Pg	28	13.8	1.1	21	12.0	1.1
Savara & Tracy ('67)	Cn-Pg	52	13.6	1.8			
Tracy & Savara ('66)	Cn-Pg				50	11.6	0.9
Ekström ('82)	Gn	53	13.6	0.7	50	12.2	1.1
Bambha ('61)	Facial	25	14.3	1.2	25	12.1	1.0
Present study	S-Gn	104	14.3	0.8	105	12.1	0.4
Somatic growth							
Tanner et al. ('76)	Stature	55	13.9	0.8	35	11.9	0.9
	Sitting height	55	14.2	0.9	35	12.2	1.0
Preece & Baines ('78)	Stature	35	14.2	0.9	23	11.9	0.7
Largo et al. ('78)	Stature	112	13.9	0.8	110	12.2	1.0
Preece & Heinrich ('81)	Stature	61	13.9	1.0	38	11.7	0.7
Gasser et al. ('84)	Stature	45	13.9	1.0	45	12.2	0.8

[1]Estimated from figures.
[2]First pubertal spurts.

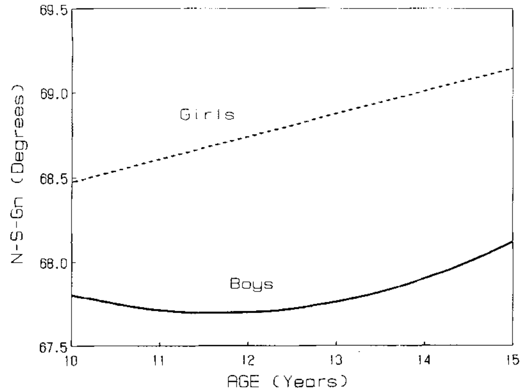

Fig. 4. Growth direction of gnathion (N-S-Gn) from 10 to 15 years of age.

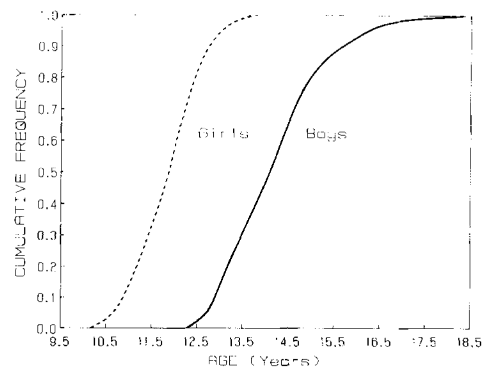

Fig. 6. Cumulative frequency plot for the age of peak pubertal velocity for sella-gnathion (S-Gn).

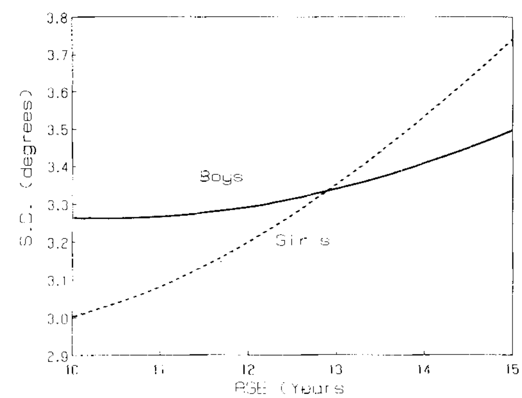

Fig. 5. Standard deviations from 10 to 15 years of age for nasion-sella-gnathion (N-S-Gn).

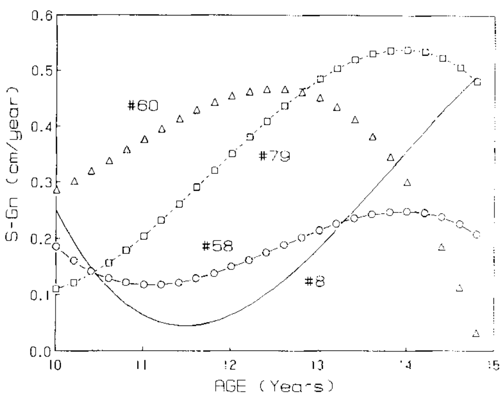

Fig. 7. Velocity curves for four boys showing normal variation in peak adolescent velocity.

By simulating multivariate normal distributions for the random variables (Goldstein, 1986a) based on their parameter estimates, cumulative probability functions of the various growth "events" for boys and girls can also be derived. The 5th and 95th percentiles for the age of maximum pubertal velocity are 12.9 and 16.5 years for boys and 10.9 and 13.0 years for girls (Fig. 6). The distributions are positively skewed for boys and leptokurtotic for both boys and girls. Similarly, the 5th and 95th percentiles for the age of minimum prepubertal velocity for boys are 8.6 and 11.8 years, respectively, with nonzero skewness and kurtosis. Goldstein (1986a) recently reported nonnormal distributions for statural growth. It is evident that boys experiencing later adolescent maximum velocities were not included in the sample. As such, the median age of 14.3 years derived from the simulation probably more closely reflects the actual age of the pubertal peak.

In conclusion, multilevel procedures model craniofacial growth effectively and efficiently. Due to the sample sizes used and the correlated nature of serial growth measures, the estimated growth rates and variances might be expected to be more precise than those that would have been obtained by using conventional procedures, i.e., cross-sectional descriptions and yearly increments (Goldstein, 1979). Fitting ordinary polynomials would have required eliminating 33% of the sample and biasing variance estimates by adjusting measurement occasion. Moreover, the observed correlations between parameter estimates serve as a basis for estimating various characteristics of individuals' growth curves when only limited data points are available. Since clinicians often use pubertal growth to maximize treatment effects and minimize time requirements, the findings hold important practical implications. They provide the information necessary to generate longitudinal growth standards from the fixed and random parameters, thereby making it possible to more accurately evaluate, and even predict, an individual's growth changes at gnathion.

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APPENDIX A: STATISTICAL METHODOLOGY

The within-subject model

The growth changes for the cephalometric landmark gnathion are related to polynomials in age as follows:

$$Y_{it} = \sum_j \beta_{ij} X_t^j + \epsilon_{it},\ t = 1 \ldots n_i \quad \{1\}$$

where

n_i = number of measurement occasions for subject *i*,
x_t = age at occasion *t*,
j = index of the polynomial coefficients,
j (0, . . . ,4) for sella-gnathion of boys,
j (0, . . . ,3) for sella-gnathion of girls,
j (0,1,2) for nasion-sella-gnathion of boys,
and
j (0,1) for nasion-sella-gnathion of girls.
Y_{it} = estimate for subject *i* at occasion *t*.

The model estimates the mean value of Y at occasion *t*, with ϵ_{it} being the random variable describing the residual-from the value predictions by the rest of the model-for the thchild at the thmeasurement occasion. The expected value of ϵ_{it}, $E(\epsilon_{it}) = 0$ with a variance of var $(\epsilon_{it}) = \sigma_0{}^2$.

The between-subject model

In order to focus on the mean coefficients, {1} can be extended and rewritten as:

$$Y_{it} = \beta_0 + \beta_1 X_t + \beta_2 X_t^2 + \ldots + \beta_j X_t J + \delta_{it}$$

where $\beta_0, \beta_1, \ldots, \beta_j$ are the expected values of $\beta_{i0}, \beta_{i1}, \ldots, \beta_{ij}$. The δ_{it} now incorporates individual coefficients and

$$\delta_{it} = (\beta_{i0} - \beta_0) + (\beta_{i1} - \beta_1)X_t + \ldots + (\beta_{ij} + \beta_j)X_t^j + \epsilon_{it}.$$

The ϵ_{it} are independent within an individual and the terms $(\beta_{i0} - \beta_0) + (\beta_{i1} - \beta_1)X_t + \ldots + (\beta_{ij} + \beta_j)X_t^j$ are common to all the δ_{it} within an individual. When $\beta_{0i}, \beta_{i1}, \ldots, \beta_{ij}$ are higher than the means, they will generate values higher than the average. Thus in

$$\beta_{ij} = \beta_j + \gamma_{ij} \quad \{2\}$$

the γ_{ij} are not necessarily independent, with $E(\gamma_{ij}) = 0$ and covariance $(\gamma_{ij}, \gamma_{ij}) = U = \{\sigma_{\mu,jj}\}$. U, which represents the variance-covariance matrix of the individual polynomial growth curve coefficients, can be represented as:

$$\begin{array}{cccccc} & \beta_0 & \beta_1 & \beta_2 & \cdot \quad \cdot & \beta_j \\ \beta_0 & \sigma^2{}_{\mu,0} & & & & \\ \beta_1 & \sigma_{\mu,01} & \sigma^2{}_{\mu,1} & & & \\ \beta_2 & \epsilon_{\mu,02} & \sigma_{\mu,12} & \sigma^2{}_{\mu,2} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \beta_j & \sigma_{\mu,0j} & \sigma_{u,ij} & \sigma_{\mu,2j} & \cdot \quad \cdot & \sigma^2{}_{\mu,j} \end{array}$$

Combining {1} and {2} the complete model can be written as:

$$\begin{aligned} Y_{it} &= \sum_j \beta_{ij} X_t^j + \sigma_{it} \\ \text{where} & \\ \beta_{i0} &= \beta_0 + \gamma_{i0} \\ \beta_{il} &= \beta_i + \gamma_{il} \\ \cdot & \quad \cdot \quad \cdot \\ \beta_{ij} &= \beta_i + \gamma_{ij} \end{aligned}$$

with $E(\gamma_{ij}) = E(_{it}) = 0$ and $var(\epsilon_{it}) = \sigma^2{}_0$, cov $(\gamma_{ij}, \gamma_{ij},) = U$.

More detailed procedures for estimating the within-subject model, together with U and $\sigma^2{}_0$, are given by Goldstein (1986b, 1987).

Depending on the number of terms (B_{ij}) permitted to vary randomly between subjects, the between-subjects variance at occasion *t* will be a function of the explanatory variables (X_t^j) and the general term will be given by: $X^TU\ X$, where X is a vector of explanatory variables and U is defined above. For the present analyses, for example, the polynomial coefficients were allowed to vary randomly up to the quartic term between-boys and up to the quadratic term between-girls. As such, the contribution to the between-subject variance for girls is at age *t* is given by:

$$\begin{aligned} &\sigma^2{}_{\mu,0} + 2\sigma_{\mu,10}X^t \\ &\quad + \sigma^2{}_{\mu,1}X^2t + 2\sigma_{\mu,02}\ X^2t \\ &\quad + 2\sigma_{\mu,12}\ X^3t \\ &\quad + \sigma^2{}_{\mu,2}\ X^4t + \sigma^2 0 \end{aligned} \quad \{3\}$$

The first derivative of the within-subject model (fixed part) gives the mean velocities for the growth curves and the first derivative of {3} provides the between-subjects variance for the mean velocity curves. By setting the second derivative of the fixed part of the model to zero (0), we can estimate the minimum and maximum velocities (zero acceleration). For example, the ages of preadolescent minimal and adolescent maximum velocities for boys are obtained by solving for:

$$\beta_2 + 3\beta_3\ X_t + 6\beta\ X_t^2 = 0$$