Modeling longitudinal mandibular growth: Percentiles for gnathion from 6 to 15 years of age in girls

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Growth of the cephalometric landmark gnathion is modeled mathematically with multilevel statistical techniques. The findings, pertaining to a mixed longitudinal sample (N = 772) of 105 girls, 6 to 15 years of age, provide the most accurate descriptions of longitudinal mandibular growth presently available. Polar, rather than rectangular, coordinates are used to better distinguish between the amount and direction of growth. The velocity curve for sella-gnathion includes growth spurts during childhood (7.5 years) and adolescence (12.7 years). Growth direction of gnathion changes regularly throughout the age range, indicating a relative increase of vertical over horizontal growth. These reference standards serve as a basis for comparing and better understanding abnormal growth patterns. (AM J ORTHOD DENTOFAC ORTHOP 1989;95:60-6.)

Polynomial regression models are well suited for studying longitudinal craniofacial growth.[1,2] They are easy to compute, can be fitted over short age ranges, do not impose a predetermined structure on the growth curve, and are able to incorporate explanatory variables other than age.[3] As such, they hold important advantages over nonlinear parametric[4,5] and nonparametric[6,7] modeling procedures. However, traditional polynomial models are not easy to use if data are missing and fixed intervals (exact ages) are required between measurement occasions, which makes them disadvantageous from a practical standpoint.

A new class of statistical models that overcomes these difficulties and holds important advantages over existing procedures has recently been introduced.[8,9] The methods make use of growth's hierarchical structure. The basic model separates variation into two parts or levels: subjects are at the higher level and measurement occasions are at the lower level. A discussion of the general model has been given by Goldstein.[3,8]

The present study investigates the *longitudinal* mandibular growth of French-Canadian girls, aged 6 to 15 years. The objectives are to (1) introduce the tremendous potential of multilevel models to the clinical community, and (2) to make available reference data necessary to perform growth assessments/predictions. It is assumed that accurate descriptions of *normative* mandibular growth are necessary to recognize abnormal growth patterns and to evaluate growth changes. Two-level polynomial models are used to estimate the means and random variation of the growth curves.

MATERIALS AND METHODS

The data are derived from serial lateral cephalograms collected by the Human Growth Research Center, Université de Montréal. They pertain to French-Canadian children drawn from three school districts in Montreal chosen to represent the different socioeconomic sectors of the larger population. The sample and sampling procedures have been previously described.[10,11] A mixed longitudinal subsample of 105 girls, followed between 6 and 15 years of age (N = 772), was selected on the basis of available and suitable serial data (Table I).

All cephalograms were traced and digitized by the same technician (L.L.). The analyses describe growth changes for the cephalometric landmark gnathion, defined as the point on the symphysis formed by bisecting the projections of the mandibular plane and a plane perpendicular to the mandibular plane and tangential to the most anterior point on the mandible. Technical reliability[12] has been estimated at 98.4% and 98.8% for the horizontal and vertical aspects of gnathion, respec-

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Table I. Sample sizes and age distributions—French-Canadian girls 6 to 15 years of age

Age group	*N*	*Median age*	*Minimum age*	*Maximum age*
6	50	5.97	5.95	6.14
7	47	6.98	6.93	7.03
8	50	7.98	7.92	8.08
9	46	8.99	8.95	9.04
10	95	9.97	9.95	10.07
11	96	10.96	10.92	11.06
12	104	11.96	11.92	12.09
13	103	12.96	12.95	13.29
14	100	13.96	13.95	14.20
15	81	14.96	14.87	15.13

tively. The measurements have been corrected for radiographic enlargement (0.8892%). The growth of gnathion was evaluated relative to the cephalometric point sella; the sella-nasion (S-N) reference plane was used for orientation. To better distinguish the *amount* and *direction* of mandibular growth from S-N, the digitized, rectangular (X,Y) coordinates were transformed into polar (τ, θ) form.

RESULTS

Growth changes in the distance sella-gnathion, hereafter referred to as gnathion length, are presented in the first set of analyses (Table II). To model the potential growth spurts during childhood and adolescence, one might expect to fit at least a fifth-order polynomial to the age range. This expectation is confirmed statistically by the fixed part of the model, relating gnathion length to a quintic polynomial in age. The fixed part describes the mean growth curve. Since age is measured from 10 years, the intercept term estimates gnathion length when age equals zero. The remaining coefficients quantify the shape of the curve between 6 and 15 years. A quintic model indicates that growth velocity changes direction four times throughout the age range, producing two points of maximum velocity separated by one point of minimum velocity (that is, three points of zero acceleration). Fig. 1, *A*, plotting the individuals' predicted values against their standardized residuals, shows no systematic deviations and substantiates the fit of the quintic polynomial.

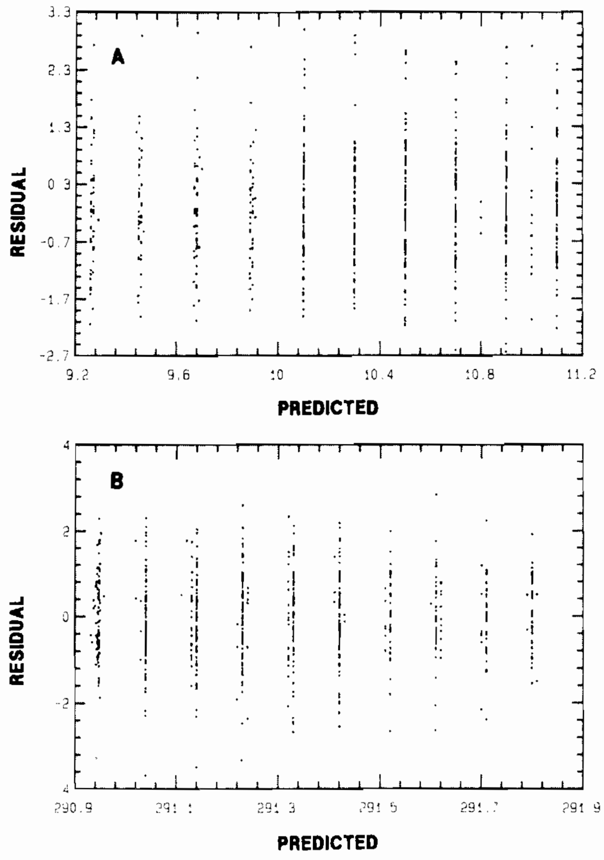

Fig. 1. Predicted values plotted against residuals for sella-gnathion in centimeters **(A)**, and nasion-sella-gnathion in degrees **(B)**.

The model's random coefficients pertain to residual variation (see Appendix I for details). Dispersion about the individuals' growth curves is described by a single within-subject (age level) variance. This indicates that there is little or no relationship between variance and age; within-subject variance remains relatively stable across the age range. The between-subject (subject level) coefficients describe the deviation of each individual's coefficients from the fixed model. In this analysis the constant, linear, and quadratic terms were allowed to vary randomly between subjects; the cubic, quartic, and quintic terms remained fixed. Random variation therefore has a three-variate distribution at the subject level. This indicates that variation about the mean growth curve follows a curvilinear pattern. Between subjects the constant and linear coefficients are positively correlated (0.29). The constant and quadratic coefficients are negatively correlated (0.46); the linear and quadratic coefficients are poorly related (0.16). The correlations indicate that large subjects tend to have a larger growth velocity (Fig. 2, *A*) and more growth deceleration (Fig. 2, *B*).

Fig. 3, integrating the foregoing results, gives the growth percentiles for gnathion length between 6 and 15 years of age. The 50th percentile, estimated from the fixed part of the model, shows that length increases approximately 18 mm during the 9-year period. Two periods of maximum velocity are indicated: one during

Table II. Growth models for gnathion—French-Canadian girls 6 to 15 years of age

Explanatory variables	*Length (cm)*		*Angle (°)*	
	Estimates	*Standard errors*	*Estimates*	*Standard errors*
Fixed coefficients				
β_0 (intercept)	10.097		291.421	
β_1 (age)	0.19578	0.0060804	−0.095214	0.03073
β_2 (age^2)	−0.00095773	0.0024418		
β_3 (age^3)	0.0031303	0.00089644		
β_4 (age^4)	0.00008889	0.00013871		
β_5 (age^5)	−0.00013953	0.000038225		
Random coefficients				
Age level				
σ_0^2	0.0079038	0.00049529	0.49151	0.029203
Subject level				
$^{\sigma^2}\mu,0$	0.15480	0.021755	8.1530	1.1473
$^{\sigma^2}\mu,1$	0.00061323	0.00013652	0.08268	0.013561
$^{\sigma^2}\mu,2$	0.000017833	0.0000064337		
$^{\sigma}\mu,01$	0.0027919	0.0013191	0.32271	0.092678
$^{\sigma}\mu,02$	−0.00073076	0.00028709		
$^{\sigma}\mu,12$	−0.000017162	0.000020982		
Correlations				
$^{\rho}\mu,01$	0.287		0.393	
$^{\rho}\mu,02$	−0.465			
$^{\rho}\mu,12$	−0.165			
Iterations	6		5	
Measurements	772		772	
Subjects	105		105	
Age measured from	10 yr		10 yr	
Relative accuracy for convergence	10^{-3}		10^{-3}	

childhood and another during adolescence. Growth begins to level off at 15 years of age.

Gnathion length may be obtained at *any* age between 6 and 15 years by solving the linear equation in Appendix I for the fixed part of the model. For example, sella-gnathion at 13 years of age (t = 13 − 10) is given by:

$$Y_3 = 10.097 + 0.19578\,(3) - 0.00095773\,(3^2) + 0.0031303\,(3^3) + 0.00008889\,(3^4) - 0.00013953\,(3^5)$$

Mean growth velocity at 13 years of age is obtained from the first derivative of the same equation.

The estimated yearly velocities between 6.5 and 14.5 years (Fig. 4) more clearly show the growth changes in gnathion length. Mean maximum velocity is slightly less during childhood than adolescence. The ages of maximum and minimum velocities are obtained by solving:

$$-0.00095773 + 3\,(0.0031303)\,(3) + 6\,(0.00008889)\,(3^2) + 10\,(-0.00013953)\,(3^3) = 0$$

which yields to the following estimates: 7.5 years for peak childhood velocity, 10.1 years for minimum preadolescent velocity, and 12.7 years of maximum adolescent velocity. The apparent reduction in variation is attributable to the larger samples after 10 years of age. Supplemental analyses, based on comparable sample sizes throughout the age range, substantiate that variation in growth velocity is actually greater during childhood than adolescence.

The next set of analyses pertain to growth direction as described by the angle nasion-sella-gnathion (Table II). At 10 years of age, the mean angle is approximately 291°, indicating a predominantly vertical direction of mandibular growth. The linear model fitted shows that changes in growth direction are relatively simple. The angle decreases regularly with age and the growth vector rotates posteriorly approximately 0.10° each year between the ages of 6 and 15 years. No systematic trends between the predicted values and their standardized residuals are evident (Fig. 1, *B*). The average within-subject variance is 0.49°. Variation about the mean growth (subject level) is substantially greater and again follows a curvilinear pattern. The constant and linear terms are moderately correlated (0.39), indicating that subjects with more vertical growth direc-

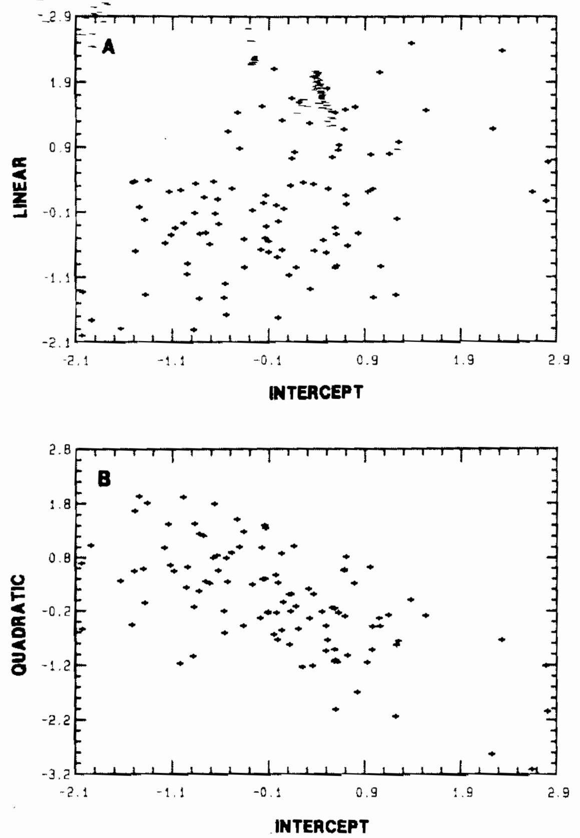

Fig. 2. Intercept plotted against linear **(A)** and quadratic **(B)** coefficients for gnathion length (sella-gnathion).

tion have greater annual rates of posterior rotation (Fig. 5).

The percentile curve (Fig. 6) shows that the mean growth direction changes slightly throughout the age range. Individuals at the higher centiles (horizontal growers) may display anterior rotation; those at the lower centiles (vertical growers) show a more substantial posterior rotation. Note should be taken of the variability between subjects in growth direction. The range of values between the 10th and 90th percentiles increases from approximately 7° at 6 years to 10° at 15 years of age. The increase of variability and differences in growth direction at the upper and lower centiles may be partially attributed to the positive association identified between the random coefficients.

DISCUSSION

Multilevel statistical models clearly offer an important tool for describing longitudinal cephalometric growth. Because of the correlated nature of the individuals' serial growth data, these models provide an accurate and parsimonious means of estimating growth rates and variances.[1-3] Conventional procedures, including cross-sectional descriptions and analyses of yearly velocities from two measurement occasions, might be expected to provide less effective and substantially more cumbersome results. Moreover, ordinary polynomial methods[1,2] would have required elimination of most of the subjects, due to missing observations, and adjustment of values to exact ages.

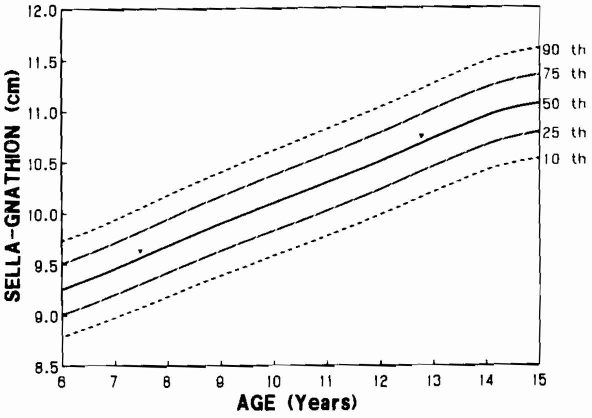

Fig. 3. Percentiles for sella-gnathion from 6 to 15 years of age in girls.

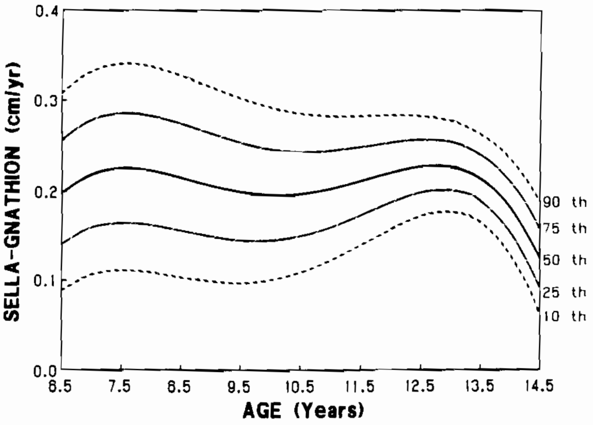

Fig. 4. Percentiles for yearly growth increments in sella-gnathion from 6.5 to 14.5 years of age in girls.

The findings leave little doubt that the growth curve for sella-gnathion includes spurts during childhood and adolescence. It simply follows the pattern previously established for statural growth. The childhood spurt for mandibular growth of girls has previously been reported between 6.5 and 7.9 years of age.[13-17] Our estimate of 7.6 years, based on a sample that is substantially larger than most others and on methods using probabilistic rather than subjective criteria, falls at the high end of the reported age range. An additional juvenile growth spurt occurring around 9 years of age, as proposed by

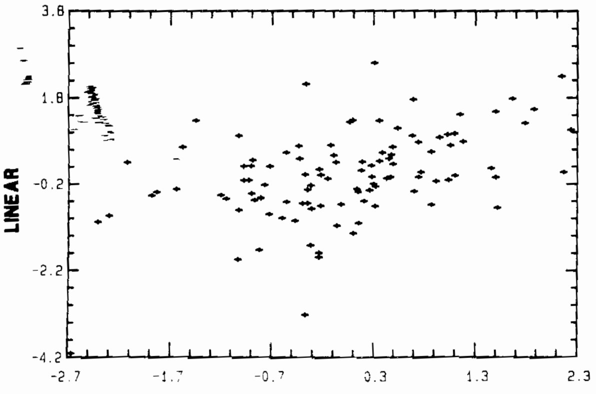

Fig. 5. Intercept plotted against linear coefficient for growth direction (nasion-sella-gnathion).

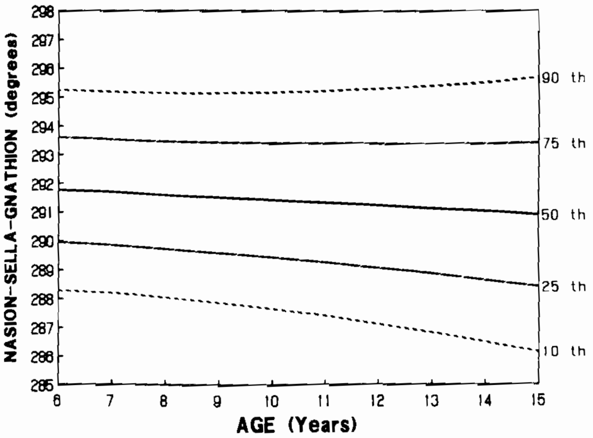

Fig. 6. Percentiles for nasion-sella-gnathion from 6 to 15 years of age in girls.

Woodside and associates[15] and Krieg,[17] is apparently not supported by the present results.

The childhood or midgrowth spurt has been established for stature[18,19] and some somatic measures.[20] With nonparametric Kernal estimation, which performs best for estimating specific events, Gasser and associates[19] reported peak height velocity for the midgrowth spurt at 7.5 years for girls. This coincides with our estimate for mandibular growth of French-Canadian subjects. Goldstein[3] gave similar results for stature with a two-level model.

The adolescent growth spurt[13,14,21-25] and the prepubertal minimum velocity[14,23,25] have been established previously for mandibular measures. The timing of these two events appears to vary across the measurements and populations studied. Peak adolescent growth velocity for girls is usually reported between 11.3 and 12.4 years of age. The duration between the ages of prepubertal minimum velocity and adolescent maximum velocity found in the present study (2.6 years) compares favorably with values reported for statural[7,26] and mandibular[6] growth.

As originally indicated, the primary objective of these analyses is to provide a global description of the mandibular growth curve for French-Canadian girls, one that efficiently summarizes all of the longitudinal changes taking place between 6 and 15 years of age. Because of sampling limitations, the model's cubic, quartic, and quintic coefficients were forced to remain fixed across subjects. As such, estimates of variation for the developmental events might be overestimated. At least a fourth-order polynomial is necessary at the subject level to accurately estimate the timing of the two peaks for each individual. Nevertheless, the rate of change in growth acceleration is $0.00313 + 0.00035t - 0.00139t^2$, which is less than zero (maximum of velocity) when t is less than 8.6 years and greater than 11.6 years. These correspond to the ages before and after which the childhood and adolescent peaks might be expected to occur. We are presently conducting further analyses, with simpler models fitted over a more restricted age range, to more precisely estimate the timing and variability for the adolescent and childhood growth spurts.

The clinical implications of the results are evident. They provide the most accurate descriptions presently available of longitudinal mandibular growth for girls between 6 and 15 years of age. As such, they may be used to evaluate growth status and future growth changes. The estimated yearly growth velocities during adolescence are particularly precise. Within-subject variability is relatively small (that is, the subjects' deviations from their growth curves remain stable across the age range). Such stability, together with the observed correlations between the random coefficients at the subject level, provide valuable information about the subjects' growth patterns. They imply that estimates of a person's growth velocity and acceleration could be predicted from a single measurement occasion. As such, the correlations serve as a basis for refining growth predictions.

Perhaps even more important, the growth curves and variation should help clinicians to better understand abnormal growth patterns. Physiologically the childhood and adolescent spurts appear to be mediated by adrenal and gonadal hormones,[27] respectively. Morphologically the growth changes at gnathion can be explained by primary and secondary displacement of the mandible, and remodeling in the symphyseal region.[28,29] Assuming that children with malocclusion follow similar growth patterns, accurate information pertaining to the two growth spurts could influence the timing and duration of orthodontic intervention.

Growth direction of gnathion changes slightly but significantly between 6 and 15 years of age. The mean angle decreases from 292° to 291°; the standard deviations increase from 2.7° to 3.8°. Further analyses are necessary to more fully appreciate these changes, particularly as they relate to changing growth patterns in the middle cranial fossa and associated changes in condylar growth.[28]

Lundström and Woodside,[30] with best-fit straight lines used to measure the direction of growth at gnathion, reported mean growth directions of 286° 289° for the Burlington and Ann Arbor samples, respectively. Their estimated standard deviations were two to three times larger than those derived for French-Canadian girls. The estimates are less precise because the standard deviations are directly proportional to the rate of change in the angle. As previously indicated, the between-subject standard deviation for the rate of change is approximately 0.3°; the within-subject rate of change is only 0.1° per year.

The observed changes in growth direction also are clinically significant. They imply that treatment planning could be compromised by projecting yearly increments along the Y axis, as is commonly practiced for growth predictions. This applies especially to vertical growers, who might be expected to display even greater rates of posterior rotation and who are most often in need of comprehensive orthodontic treatment.[31] Moreover, the observed posterior rotation indicates that the relative amount of vertical over horizontal growth increases with age. Clinicians hoping to "make use" of a child's horizontal growth potential to reduce or eliminate anteroposterior discrepancies for Class II malocclusion might consider initiating treatment earlier than is commonly accepted.

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Appendix I. Statistical methods

1. The within-subject model:

Gnathion length (sella-gnathion) and growth direction (nasion-sella-gnathion) are related to basic quintic and linear polynomials in age, respectively, as follows:

$$Y_{it} = \sum_j \beta_{ij} X_t^j + \epsilon_{it},\ t = 1 \ldots n_i\ \{\mathbf{1}\}$$

where: n_i = number of measurement occasions for subject *i*,
X_t = age at occasion *t*,
j = indexes the polynomial coefficients,
j (0, . . . ,5) for gnathion length
j (0,1) for the angle of growth direction
Y_{it} = length or angle of subject *i* at occasion *t*.

The model estimates the mean value of Y (length or angle) at occasion *t*, with ϵ_{it} being the random variable describing the residual of the i^{th} child at the t^{th} measurement occasion from the value predicted by the rest of the model. The expected value of ϵ_{it}, $E(\epsilon_{it}) = 0$ with a variance of var $(\epsilon_{it}) = \sigma_0^2$.

2. The between-subject model:

If we extend {**1**} and focus on the mean coefficients, we can rewrite {**1**} as:

$$Y_{it} = \beta_0 + \beta_1 X_t + \beta_2 X_t^2 + \ldots + \beta_j X_t^j + \delta_{it}$$

where $\beta_0, \beta_1, \ldots, \beta_j$ are the expected values of $\beta_{i0}, \beta_{i1}, \ldots, \beta_{ij}$. The δ_{it} now incorporate individual coefficients and,

$$\delta_{it} = (\beta_{i0} - \beta_0) + (\beta_{i1} - \beta_1)X_t + \ldots + (\beta_{ij} + \beta_j)X_t^j + \epsilon_{it}$$

Although it is clear that ϵ_{it} are independent from each other within an individual, the term $(\beta_{i0} - \beta_0) + (\beta_{i1} - \beta_1)X_t + \ldots + (\beta_{ij} + \beta_j)X_t^j$ are common to all the δ_{it} within an individual and if, for example, $\beta_{i0}, \beta_{i1}, \ldots, \beta_{ij}$ are higher than the means, they will generate values higher than the average. Hence, if we write

$$B_{ij} = \beta_j + \gamma_{ij}\ \{\mathbf{2}\},$$

the γ_{ij} are not necessarily independent and we have $E(\gamma_{ij}) = 0$ and covariance $(\gamma_{ij}, \gamma_{ij'}) = \mathbf{U} = \{\sigma_{\mu.jj'}\}$. Thus, **U** represents the variance-covariance matrix of the individual polynomial growth curve coefficients. The elements of **U** can be represented as:

$$\begin{array}{ccccccc}
 & \beta_0 & \beta_1 & \beta_2 & \bullet & \bullet & \beta_j \\
\beta_0 & \sigma^2_{\mu,0} & & & & & \\
\beta_1 & \sigma_{\mu,01} & \sigma^2_{\mu,1} & & & & \\
\beta_2 & \sigma_{\mu,02} & \sigma_{\mu,12} & \sigma^2_{\mu,2} & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & & \\
\beta_j & \sigma_{\mu,0j} & \sigma_{\mu,1j} & \sigma_{\mu,2j} & \bullet & \bullet & \sigma^2_{\mu,j}
\end{array}$$

Combining {**1**} and {**2**} the complete model can be written as:

$$\begin{gathered}
Y_{it} = \sum_j \beta_{ij} X_t^j + ;\epsilon_{it},\ \text{where} \\
\beta_{i0} = \beta_0 + \gamma_{i0}, \\
\beta_{i1} = \beta_1 + \gamma_{i1}, \\
\bullet \quad \bullet \quad \bullet \\
\beta_{ij} = \beta_i + \gamma_{ij}, \\
\text{with } E(\gamma_{ij}) = E(\epsilon_{it}) = 0 \text{ and } \mathrm{var}(\epsilon_{it}) = \sigma^2_0, \\
\mathrm{cov}(\gamma_{ij}, \gamma_{ij'}) = \mathbf{U}.
\end{gathered}$$

The detailed procedures to estimate the within-subject model together with **U** and σ^2_0 are given by Goldstein.[8,32]

Depending on the number of terms (β_{ij}) allowed to vary randomly between subjects, the between-subject variance at occasion *t* will be a function of the explanatory variables (X_t^j) and the general term will be given by: $\mathbf{X}^T\mathbf{U}\mathbf{X}$, where **X** is a vector of explanatory variables and **U** is as defined above. For the present analyses, the polynomial coefficients up to the quadratic term were allowed to vary randomly between subjects for length, while only the constant and linear terms were allowed to vary for the angle of growth direction. The contribution to the between-subject variance at age *t* is then:

$$\sigma^2_{\mu.0} + 2\sigma_{\mu.10}X_t + \sigma^2_{\mu.1}X^2t + 2\sigma_{\mu.02}X^2t + 2\sigma_{\mu.12}X^3_t + \sigma^2_{\mu.2}X^2_t + \sigma^2_0\ \{\mathbf{3}\}$$

for gnathion length

$$\boldsymbol{\sigma}^2_{\mu.0} + 2\sigma_{\mu.01}X_t + ;\sigma^2_{\mu.1}X_t^2 + \sigma^2_0\ \{\mathbf{4}\}$$

for growth direction

The first derivative of the within-subject model (fixed part) will give the mean velocities for the growth curves and the first derivatives of {**3**} and {**4**} will give the between-subject variances for the mean velocity curves:

$$2\sigma_{\mu.01} + \sigma^2_{\mu.1}X_t + 4\sigma_{\mu.02}X_t + 6\sigma_{\mu.12}X^2_t + 4\sigma^2_{\mu.2}X^3_t \text{ for length,}$$

and

$$2\sigma_{\mu.01} + 2\sigma^2_{\mu.1}X_t \text{ for the angle.}$$