Multilevel Factor analysis models for continuous and discrete data.

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Abstract

A very general class of multilevel factor analysis and structural equation models is proposed which are derived from considering the concatenation of a series of building blocks that use sets of factor structures defined within the levels of a multilevel model. An MCMC estimation algorithm is proposed for this structure to produce parameter chains for point and interval estimates. We show how traditional models for binary response factor analysis can be extended to fit multiple factors within a multilevel data structure. It is shown how a probit link function has useful interpretations and in particular that this allows the joint modeling of binary, ordered and continuous response variables. The model is applied to the study of country differences in a large scale study of Mathematics achievement in schools.

Keywords

Factor analysis, International comparisons, Item Response Model, Markov Chain Monte Carlo, Mixed response model, Multilevel model, Structural Equation Model.

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1. Multilevel models

The technique of multilevel modelling is now well established and has been incorporated into most of the standard statistical packages. It is convenient for present purposes to consider educational data that exhibits a hierarchical structure of students nested within classrooms, nested within schools; a 3-level structure. Other examples are repeated measures data with occasions nested within subjects or surveys with people nested within households, both of these being 2-level data structures. More complex data structures such as cross classifications and 'multiple membership structures' are extensions for which models have been developed. Goldstein (2003) provides a detailed exposition with references to further application areas.

A general model for the 3-level schooling case, assuming Normality can be written as

$$y_{ijk} = (X\beta)_{ijk} + Z^{(3)}_{ijk} w_k + Z^{(2)}_{ijk} v_{jk} + e_{ijk}$$
$$w \sim MVN(0,\Omega_w), \quad v \sim MVN(0,\Omega_v)$$

where the superscripts indicate the level, (1)

and the more general $n-$level model can be written compactly as

$$Y = X\beta + Zu + e, \quad u \sim MVN(0,\Omega_u), \quad e \sim N(0,\sigma^2_e I)$$

For a generalized linear model, say with a binary response, we correspondingly have

$$\begin{aligned} g(\pi) &= X\beta + Zu, \quad u \sim MVN(0,\Omega_u) \\ Y &\sim Bernoulli(1,\pi) \end{aligned} \tag{2}$$

where g is a suitable link function and $X\beta$ refers to the fixed coefficient (regression) component of the model. We shall assume that the level 1 residual matrix is diagonal, and this will apply to our factor models too.

First, we review briefly the traditional approach to estimating these models based upon maximum likelihood.

2. Maximum likelihood estimation

For the Normal model the standard (twice) the log likelihood is

$$\begin{aligned} 2L(\Omega,\beta) &= -\log|V| - tr(V^{-1}S) = -\log|V| - (Y-X\beta)^T V^{-1}(Y-X\beta) \\ S &= (Y-X\beta)(Y-X\beta)^T, \quad V = \text{cov}(Y\,|\,X\beta) \end{aligned} \tag{3}$$

where Ω is the set of random parameters comprising the variances and covariances in (1). If we have an ML estimate of β then

$$2L(\Omega,\hat{\beta}) = -\log|V| - tr(V^{-1}S) = -\log|V| - (Y-X\hat{\beta})^T V^{-1}(Y-X\hat{\beta}) \tag{4}$$

is the profile likelihood for the random parameters Ω. A convenient algorithm known as Iterative Generalised Least Squares (IGLS) alternates between maximising (4) and then obtaining the conditional ML (GLS) estimate of Ω until convergence.

We can write the extended likelihood, referred to in different contexts as a penalised likelihood or an h-likelihood (Lee and Nelder, 2001), that includes the actual random effects (residuals) as parameters

$$2L(\Omega,\beta,u) = -\log|R| - (Y - X\beta - Zu)^T R^{-1}(Y - X\beta - Zu)$$
$$-\log|\Omega_u| - u^T\Omega_u^{-1}u \quad (5)$$
$$R = \sigma_e^2 I$$

If we maximise (5) for the random effects, given (β,Ω), we obtain the usual estimator which can be written conveniently as

$$\hat{u} = (Z^T R^{-1} Z + \Omega_u^{-1})^{-1} Z^T (Y - X\beta) \quad (6)$$

Given Ω, u the profile likelihood for the fixed effects is thus

$$2L(\hat{\Omega},\hat{u}) = -\log|\hat{R}| - (Y - X\beta - Z\hat{u})^T \hat{R}^{-1}(Y - X\beta - Z\hat{u})$$
$$-\log|\hat{\Omega}_u| - \hat{u}^T\hat{\Omega}_u^{-1}\hat{u} \quad (7)$$

So that a convenient modification of the IGLS procedure is to iterate between calculating the fixed effects using (7) which, when *R* is diagonal, is just OLS, calculating the random effects from (6) and then the random parameters using the same step as is used in the standard IGLS algorithm.

Expression (3) is known as the *marginal* log likelihood since it is obtained by integrating out the random effects regarded as 'nuisance' parameters.

Since the random effects depend on the random but not the fixed parameters, more generally we can write

$$\log[L(\beta,\Omega,U)] = \sum\{\log[f(Y|U;\beta)] + \log[f(U;\Omega)]\} \quad (8)$$

The marginal likelihood is thus given by

$$L(\beta,\Omega) = \int f(Y|U;\beta) f(U;\Omega) dU \quad (9)$$

where the first term on the right hand side is the distribution function for the responses conditional on the random effects, or residuals, *U*. The second term is the distribution function for the random effects. The first term, given *U*, depends only on the unknown parameters β and the second only on the unknown parameters Ω. Thus, for example, for a 2-level logistic binary response model where the random effects are assumed to be multivariate normal we have, since the random effects are independent across units,

$$L(\beta,\Omega) = \prod_j \int \prod_i \{(\pi_{ij})^{s_{ij}}(1-\pi_{ij})^{n_{ij}-s_{ij}}\}\Phi(u_j;\Omega)du_j$$
$$\pi_{ij} = \{1 + \exp(-X_{ij}\beta_j)\}^{-1}, \quad \beta_j = \beta + u_j \quad (10)$$

where Φ is the multivariate normal density function for the u_j and n_{ij}, s_{ij} are the numbers of trials and successes respectively.

where $\Phi(u_j;\Omega)$ is the multivariate Normal density and (10) can be written in the form $\int_{-\infty}^{\infty} P(u_j)\Phi(u_j)du_j$.

Gauss-Hermite quadrature approximates an integral such as the above as

$$\int_{-\infty}^{\infty} P(v)e^{-v^2} dv \approx \sum_{q=1}^{Q} P(x_q)w_q \quad (11)$$

where the right hand side is a Gauss-Hermite polynomial evaluated at a series of quadrature points indexed by q. Hedeker and Gibbons (1994) give a detailed discussion and also consider the multicategory (multinomial) response case. This function is then maximised using a suitable search procedure over the parameter space. Rabe-Hesketh et al., (2002) use quadrature to fit general multilevel structural equation models wit a variety of link functions. An alternative to quadrature is to use simulated maximum likelihood that is attractive for models with large numbers of random parameters; see Goldstein (2003, Appendix 4.2).

We now look at multilevel factor models. We shall briefly refer to the maximum likelihood analysis of multilevel factor analysis models and move on to develop an alternative approach using Markov Chain Monte Carlo (MCMC) estimation.

3. A multilevel factor model

We begin by considering a simple single level factor model for continuous responses, which we write as

$$\begin{aligned} &y_{ri} = \lambda_r v_i + e_{ri}, \ r = 1,...,R, \ i = 1,...,N \\ &v_i \sim N(0,1), \ e_{ri} \sim N(0,\sigma^2_{er}) \end{aligned} \quad (12)$$

where r indexes the responses and i indexes individuals. This can in fact be viewed as a 2-level model with a single level 2 random effect (v_i) with variance constrained to 1 and R level 1 units for each level 2 unit, each with their own (unique) variance.

If we knew the values of the 'loadings' λ_r then we could fit (12) directly as a 2-level model with the loading vector as the explanatory variable for the level 2 variance which is constrained to be equal to 1; if there are any measured covariates in the model their coefficients can be estimated at the same time. Conversely, if we knew the values of the random effects v_i, we could estimate the loadings; this would now be a single level model with each response variate having it's own variance. These considerations suggest that an EM algorithm can be used in the estimation where the random effects are regarded as missing data (see Rubin and Thayer, 1982). They also motivate the use of MCMC estimation that we will discuss below.

We now add a second level with its own factor structure and write

$$\begin{aligned} &Y = \Lambda_2 v_2 + u + \Lambda_1 v_1 + e \\ &Y = \{y_{rij}\}, \ u = \{u_r\}, \quad e = \{e_r\} \\ &r = 1,...,p \qquad i = 1,...,n_j \qquad j = 1,...,J \\ &v_2 \sim N(0,1), \quad v_1 \sim N(0,1), \ e_r \sim N(0,\sigma^2_{er}), \ u_r \sim N(0,\sigma^2_{ur}) \end{aligned} \quad (13)$$

where the 'uniquenesses' u (level 2) , e (level 1) are mutually independent and there are p response measures. The Λ_1, Λ_2 are the loading matrices for the level 1 and level 2 factors and the v_1, v_2 are the, independent, factor vectors at level 1 and level 2. Note that we can have different numbers of factors at each level. We adopt the convention of regarding the measurements themselves as constituting the lowest level of the hierarchy so that equation (13) is regarded as a 3-level model. Extensions to more levels are straightforward.

We can write the Normal log-likelihood for (12) as

$$\begin{aligned} 2L(\Omega,\beta,v) &= -\log|R| - (Y-\Lambda\ v\ )^T R^{-1}(Y-\Lambda\ v\ ) \\ &\quad -\log|\Omega_v| - v^T\Omega_v^{-1}v \\ \Lambda &= (\Lambda_1\ \Lambda_2)\quad v^T = (v_1\ v_2) \\ R &= diag(\sigma_{e1}^2,....\sigma_{ep}^2) \end{aligned} \tag{14}$$

with corresponding expressions for other link functions. A general approach to estimation is to form the marginal likelihood as described above. McDonald and Goldstein (1989) provide an explicit computational algorithm for the Normal response model (13). Longford and Muthen (1992) develop this approach. The latter authors, together with Goldstein (1995, Chapter 11) and Rowe and Hill (1997, 1998) also point out that consistent estimators can be obtained from a 2-stage process as follows. A 2-level multivariate response linear model is fitted using an efficient procedure such as maximum likelihood. This can be accomplished, for example as pointed out earlier by defining a 3-level model where the lowest level is that of the response variables (see Goldstein, 1995, Chapter 8 and model (15) below). This analysis will produce estimates for the (residual) covariance matrices at each level and each of these can then be structured according to an underlying latent variable model in the usual way. By considering the two matrices as two 'populations' we can also impose constraints on, say, the loadings using an algorithm for simultaneously fitting structural equations across several populations.

The present chapter describes a general approach to the estimation of such multilevel factor analysis models using Markov Chain Monte Carlo (MCMC). In the standard multilevel model (1) described above, MCMC treats the random effects at higher levels as parameters alongside the fixed coefficients, variances and covariances. The algorithm proceeds in steps where at each step a parameter or set of parameters is updated by sampling from the distribution for those parameters conditional on the current values for all the other parameters, the data and the prior distributions. For each parameter this results in a chain of correlated values that, after the chain has become stationary, can be used for inference. Thus, the mean and standard deviation provide estimates corresponding to the traditional maximum likelihood estimate and its standard error. An advantage of MCMC is that, given a long enough chain, we can obtain exact interval estimates based upon quantiles rather than relying on large sample approximations.

We now describe the details of an MCMC algorithm for the factor analysis model.

4. MCMC estimation for the factor analysis model

We first develop our MCMC algorithm for the multivariate Normal model. This is followed by an extension to the binary and mixed response case, where we also give a detailed example. Further discussion of the multivariate Normal model can be found in Goldstein and Browne (2002).

To show the steps of the MCMC algorithm we write (13) in the more detailed form

$$y_{rij} = \beta_r + \sum_{f=1}^{F} \lambda_{fr}^{(2)} \nu_{fj}^{(2)} + \sum_{g=1}^{G} \lambda_{gr}^{(1)} \nu_{gij}^{(1)} + u_{rj} + e_{rij}$$
$$u_{rj} \sim N(0, \sigma_{ur}^2), e_{rij} \sim N(0, \sigma_{er}^2), \nu_{fj}^{(2)} \sim MVN_F(0, \Omega_2), \nu_{gij}^{(1)} \sim MVN_G(0, \Omega_1) \qquad (15)$$
$$r = 1, \ldots, R, i = 1, \ldots, n_j, j = 1, \ldots, J, \sum_{j=1}^{J} n_j = N$$

Again we have R responses for N individuals split between J level 2 units. We have F sets of factors, $\nu_{fj}^{(2)}$ defined at level 2 and G sets of factors, $\nu_{gij}^{(1)}$ defined at level 1. We also introduce the fixed part of the model but for simplicity restrict our algorithm to a single intercept term β_r for each response although it is easy to extend the algorithm to arbitrary fixed terms. The residuals at levels 1 and 2, e_{rij} and u_{rj} are assumed to be independent.

Although this allows a very flexible set of factor models it should be noted that in order for such models to be identifiable suitable constraints must be put on the parameters. See Everitt (1984) for further discussion of identifiability. These will consist of fixing the values of some of the elements of the factor variance matrices, Ω_1 and Ω_2 and/or some of the factor loadings, $\lambda_{fr}^{(2)}$ and $\lambda_{gr}^{(1)}$.

The algorithms presented will give steps for all parameters and so any parameter that is constrained will simply maintain its chosen value and will not be updated. We will initially assume that the factor variance matrices, Ω_1 and Ω_2 are known (completely constrained) and then discuss how the algorithm can be extended to encompass partially constrained variance matrices. The parameters in the following steps are those available at the current iteration of the algorithm.

4.1 Prior Distributions

For the algorithm we will assume the following general priors

$$p(\beta_r) \sim N(\beta_r^*, \sigma_{br}^2)$$
$$p(\lambda_{fr}^{(2)}) \sim N(\lambda_{fr}^{(2)*}, \sigma_{2fr}^2), p(\lambda_{gr}^{(1)}) \sim N(\lambda_{gr}^{(1)*}, \sigma_{1gr}^2)$$
$$p(\sigma_{ur}^2) \sim \Gamma^{-1}(a_{ur}^*, b_{ur}^*), p(\sigma_{er}^2) \sim \Gamma^{-1}(a_{er}^*, b_{er}^*)$$

4.2 Known factor variance matrices

We assume that the factor variance matrices are known so that we can use a Gibbs sampling algorithm which will involve updating parameters in turn by generating new values from the following 8 sets of conditional posterior distributions.

Step 1: Update current value of β_r (*r=1,…,R*) from the following distribution:

$$p(\beta_r) \sim N\left(D_{br} \left(\frac{\sum_{ij} d_{rij}^{\beta}}{\sigma_{er}^2} + \frac{\beta_r^*}{\sigma_{br}^2} \right), D_{br} \right) \text{ where}$$

$$D_{br} = \left(\frac{N}{\sigma_{er}^2} + \frac{1}{\sigma_{br}^2} \right)^{-1} \text{ and } d_{rij}^{\beta} = e_{rij} + \beta_r .$$

Step 2: Update $\lambda_{fr}^{(2)}$ (*r=1,…,R, f =1,…,F* where not constrained) from the following distribution :

$$p(\lambda_{fr}^{(2)}) \sim N\left(D_{fr}^{(2)} \left(\frac{\sum_{ij} v_{fj}^{(2)} d_{rijf}^{(2)}}{\sigma_{er}^2} + \frac{\lambda_{fr}^{(2)*}}{\sigma_{2fr}^2} \right), D_{fr}^{(2)} \right)$$

where $D_{fr}^{(2)} = \left(\frac{\sum_j n_j (v_{fj}^{(2)})^2}{\sigma_{er}^2} + \frac{1}{\sigma_{2fr}^2} \right)^{-1}$ and $d_{rijf}^{(2)} = e_{rij} + \lambda_{fr}^{(2)} v_{fj}^{(2)}$.

Step 3: Update $\lambda_{gr}^{(1)}$ (*r=1,…,R, g =1,…,G* where not constrained) from the following distribution :

$$p(\lambda_{gr}^{(1)}) \sim N\left(D_{gr}^{(1)} \left(\frac{\sum_{ij} v_{gij}^{(1)} d_{rijg}^{(1)}}{\sigma_{er}^2} + \frac{\lambda_{gr}^{(1)*}}{\sigma_{1gr}^2} \right), D_{gr}^{(1)} \right)$$

where $D_{gr}^{(1)} = \left(\frac{\sum_{ij} (v_{gij}^{(1)})^2}{\sigma_{er}^2} + \frac{1}{\sigma_{1gr}^2} \right)^{-1}$ and $d_{rijg}^{(1)} = e_{rij} + \lambda_{gr}^{(1)} v_{gij}^{(1)}$.

Step 4: Update $v_j^{(2)}$ (*j= 1,…,J*) from the following distribution:

$$p(v_j^{(2)}) \sim MVN_F\left(D_j^{(2)} \left(\sum_r \sum_{i=1}^{n_j} \frac{\lambda_r^{(2)} d_{rij}^{(2)}}{\sigma_{er}^2} \right), D_j^{(2)} \right)$$

where $D_j^{(2)} = \left(\sum_r \frac{n_j \lambda_r^{(2)} (\lambda_r^{(2)})^T}{\sigma_{er}^2} + \Omega_2^{-1} \right)^{-1}$ and

$$d_{rij}^{(2)} = e_{rij} + \sum_{f=1}^{F} \lambda_{fr}^{(2)} v_{fj}^{(2)}, \ \ \lambda_r^{(2)} = (\lambda_{1r}^{(2)}, \ldots \lambda_{Fr}^{(2)})^T, \ \ v_j^{(2)} = (v_{1j}^{(2)}, \ldots, v_{Fj}^{(2)})^T .$$

Step 5: Update $v_{ij}^{(1)}$ (*i=1,…,n*$_j$*, j= 1,…,J*) from the following distribution:

$$p(v_{ij}^{(1)}) \sim MVN_G\left(D_{ij}^{(1)} \left(\sum_r \frac{\lambda_r^{(1)} d_{rij}^{(1)}}{\sigma_{er}^2} \right), D_{ij}^{(1)} \right)$$

where

$$D_{ij}^{(1)} = \left(\sum_r \frac{\lambda_r^{(1)} (\lambda_r^{(1)})^T}{\sigma_{er}^2} + \Omega_1^{-1} \right)^{-1} \text{ and}$$

$$d_{rij}^{(1)} = e_{rij} + \sum_{g=1}^{G} \lambda_{gr}^{(1)} v_{gj}^{(1)}, \ \ \lambda_r^{(1)} = (\lambda_{1r}^{(1)}, \ldots \lambda_{Gr}^{(1)})^T, \ \ v_{ij}^{(1)} = (v_{1ij}^{(1)}, \ldots, v_{Gij}^{(1)})^T .$$

Step 6: Update u_{rj} (*r=1,...,R, j=1,...,J*) from the following distribution :

$$p(u_{rj}) \sim N\left(\frac{D_{rj}^{(u)}}{\sigma_{er}^2} \sum_{i=1}^{n_j} d_{rij}^{(u)}, D_{rj}^{(u)} \right)$$

where

$$D_{rj}^{(u)} = \left(\frac{n_j}{\sigma_{er}^2} + \frac{1}{\sigma_{ur}^2} \right)^{-1} \text{ and } d_{rij}^{(u)} = e_{rij} + u_{rj} .$$

Step 7: Update σ_{ur}^2 from the following distribution: $p(\sigma_{ur}^2) \sim \Gamma^{-1}(\hat{a}_{ur}, \hat{b}_{ur})$ where

$\hat{a}_{ur} = J/2 + a_{ur}^*$ and $\hat{b}_{ur} = \frac{1}{2}\sum_j u_{rj}^2 + b_{ur}^*$.

Step 8: Update σ_{er}^2 from the following distribution: $p(\sigma_{er}^2) \sim \Gamma^{-1}(\hat{a}_{er}, \hat{b}_{er})$ where $\hat{a}_{er} = N/2 + a_{er}^*$ and $\hat{b}_{er} = \frac{1}{2}\sum_{ij} e_{rij}^2 + b_{er}^*$.

Note that the level 1 residuals, e_{rij} can be calculated by subtraction at every step of the algorithm.

4.3 Unconstrained factor covariances

In the general algorithm we have assumed that the factor variances are all constrained. Typically we will fix the variances to equal 1 and the covariances to equal 0 and have independent factors. This form will allow us to simplify steps 4 and 5 of the algorithm to univariate Normal updates for each factor separately. We may however wish to consider correlations between the factors. Here we will modify our algorithm to allow another special case where the variances are constrained to be 1 but the covariances can be freely estimated. Where the resulting correlations obtained are estimated to be close to 1 or –1 then we may be fitting too many factors at that particular level. As the variances are constrained to equal 1 the covariances between factors equal the correlations between the factors. This means that each covariance is constrained to lie between –1 and 1. We will consider here only the factor variance matrix at level 2 as the step for the level 1 variance matrix simply involves changing subscripts. We will use the following priors:

$$p(\Omega_{2,lm}) \sim Uniform(-1,1) \forall l \neq m$$

Here $\Omega_{2,lm}$ is the *l,m*-th element of the level 2 factor variance matrix. We will update these covariance parameters using a Metropolis step and a Normal random walk proposal (see Browne (2003) for more details on using Metropolis Hastings methods for constrained variance matrices).

Step 9: At iteration t generate $\Omega_{2,lm}^* \sim$ N($\Omega_{2,lm}^{(t-1)}, \sigma_{plm}^2$) where σ_{plm}^2 is a proposal distribution variance that has to be set for each covariance. Then if $\Omega_{2,lm}^* > 1$ or $\Omega_{2,lm}^*$

< -1 set $\Omega_{2,lm}^{(t)} = \Omega_{2,lm}^{(t-1)}$ as the proposed covariance is not valid else form a proposed new matrix Ω_2^* by replacing the *l,m* th element of $\Omega_2^{(t-1)}$ by this proposed value. We then set

$\Omega_{2,lm}^{(t)} = \Omega_{2,lm}^{*}$ with probability $\min(1, p(\Omega_2^* \mid \nu_{fj}^{(2)}) / p(\Omega_2^{(t-1)} \mid \nu_{fj}^{(2)})$ and $\Omega_{2,lm}^{(t)} = \Omega_{2,lm}^{(t-1)}$ otherwise.

Here $p(\Omega_2^* \mid \nu_{fj}^{(2)}) = \prod_j |\Omega_2^*|^{-1/2} \exp((\nu_{fj}^{(2)})^T (\Omega_2^*)^{-1} \nu_{fj}^{(2)})$ and

$$p(\Omega_2^{(t-1)} \mid \nu_{fj}^{(2)}) = \prod_j |\Omega_2^{(t-1)}|^{-1/2} \exp((\nu_{fj}^{(2)})^T (\Omega_2^{(t-1)})^{-1} \nu_{fj}^{(2)})$$

This procedure is repeated for each covariance that is not constrained.

4.4 Missing Data

Where some of the responses are missing this poses no problem for the MCMC methods if we are prepared to assume missingness is at random or effectively so by design. This is equivalent to giving the missing data a uniform prior. We then have to simply add an extra Gibbs sampling step to the algorithm to sample the missing values at each iteration. As an illustration we will consider an individual who is missing response *r*. In a factor model the correlation between responses is explained in the factor terms and conditional on these terms the responses for an individual are independent and so the conditional distributions of the missing responses have simple forms.

Step 10: Update y_{rij} ($r=1,\ldots,R$, $i=1,\ldots,n_j$, $j=1,\ldots,J$ $\forall$ y_{rij} that are missing) from the following distribution, given the current values, $y_{rij} \sim N(\hat{y}_{rij}, \sigma_{er}^2)$ where $\hat{y}_{rij} = \beta_r + \sum_{f=1}^{F} \lambda_{fr}^{(2)} \nu_{fj}^{(2)} + \sum_{g=1}^{G} \lambda_{gr}^{(1)} \nu_{gij}^{(1)} + u_{rj}$.

Goldstein and Browne (2002) discuss the extension of this model to the general structural equation case.

5. Binary response factor models

Modelling of data that consist of binary or ordered responses to questions in an achievement or similar test instrument has a long history. Goldstein and Wood (1988) describe the history of mental testing from the early work of Lawley (1942) and the work of Lord and Novick (1968) on item response models to more recent developments of general factor analysis modelling (Bock et al, 1988). Rod McDonald has made important contributions to this area through his discussions of test item dimensionality and models for nonlinear factor analysis (McDonald, 1981, 1985).

The early work was characterised by 'fixed effect' models of the kind

$$f(\pi_{ri}) = \beta_{0r} + \beta_{1r}\theta_i \qquad (16)$$

relating the probability of a correct response to the *r-th* item for the *i*-th respondent, where typically a logit link function is used for the probability. The most common link function *f* is a logit or probit. The response, *y*, is typically (0,1) and we have the local, or conditional, independence assumption

$$y_{ij} \overset{iid}{\sim} Bin(1, \pi_{ij})$$

This is often referred to, somewhat inaccurately, as a 2-parameter model where in (16) each response is characterised by an intercept, β_{0r} and factor coefficient β_{1r}, and each respondent has a factor value θ_i. This gives rise to a model with *N+2p* parameters where *N* is the number of respondents and *p* is the number of items or questions. Extensions to the case where responses are on an ordered scale (a graded response or partial credit model (Baker, 1992)) relate the cumulative proportion of success to a linear function via a suitable link function, for example the cumulative odds model for category *h* of item *r*

$$f(\sum_{g=1}^{h} \pi_{gri} / \sum_{g=h+1}^{t_r} \pi_{gri}) = \beta_{0rh} + \beta_{1r}\theta_i, \; h = 1,...,t_r - 1 \qquad (17)$$

where t_r indexes the final category of item *r*.

Such fixed effect models have more recently been superseded by 'random effects' models (Bartholomew and Knott, 1999) where the individual parameter θ_i is assumed to have a distribution, typically Normal, across individuals. This provides both more efficient estimates and straightforward ways of handling missing responses. More importantly, it allows for the fitting of more than one parameter for individuals so that we can write down a general multidimensional binary (or ordered) extension of (16)

$$\begin{aligned} & f(\pi_{ri}) = \beta_{0r} + \sum_{h=1}^{q} \beta_{hr}\theta_{hi}, \theta_i \sim MVN(0, \Omega) \\ & y_{ri} \overset{iid}{\sim} Bin(1, \pi_{ri}) \end{aligned} \qquad (18)$$

which is simply a single level binary response factor model.

Having fitted such a model we can obtain estimates of factor values, or scores, for each individual on each factor. In practice models with more than one factor dimension have been used rarely nor, typically, are covariates incorporated, for example for gender or other predictors. We shall explore some of the consequences of this below in the analysis of a large-scale data set.

We now introduce a multilevel model, as with the Normal response case, that recognises that groups such as schools, may differ in their response probabilities. We write

$$\begin{aligned} & f(\pi_{rij}) = \beta_{0r} + \sum_{h=1}^{q_1} \beta_{hr}^{(1)}\theta_{hij}^{(1)} + \sum_{h=1}^{q_2} \beta_{hr}^{(2)}\theta_{hj}^{(2)} + u_{rj} \\ & \theta_{ij}^{(1)} \sim MVN(0, \Omega_1), \; \theta_j^{(2)} \sim MVN(0, \Omega_2) \qquad (19) \\ & y_{rjk} \overset{iid}{\sim} Bin(1, \pi_{rjk}) \end{aligned}$$

We have now added a second set of factors, indexed by the superscript (2) varying at the group level 2, independently of the individual level factors, indexed by the superscript (1). In contrast to the Normal response factor model the level 1 variance is constrained by the assumption of binomial variation and the factor structure has a nonlinear link with the responses. We shall retain the notational conventions for binary response models, generalized from (17), where we have the following equivalences for the factor structure between (18) and (15):

$$\beta_{hr}^{(1)} \equiv \lambda_{gr}^{(1)}, \quad \beta_{hr}^{(2)} \equiv \lambda_{fr}^{(2)}$$

$$\theta_{hij}^{(1)} \equiv v_{gij}^{(1)}, \quad \theta_{hj}^{(2)} \equiv v_{fj}^{(2)}$$

The level 2 residuals, u_{rj} are assumed independent $N(0,\sigma_{ur}^2)$.

We will show how to specify and fit such a model and use it with a large scale survey of student achievement.

6. Data

The data are taken from the Programme for International Student Assessment (PISA) carried out under the auspices of OECD in 2000 in 32 industrialised countries (OECD, 1999). The data sets, together with full descriptions are available on line (www.pisa.oecd.org). The full data set included a student and school questionnaire together with tests of reading, mathematics and science. A sample of 14-15 year old school students was selected in each country with a 70% response rate as a minimum requirement for inclusion. The OECD programme plans further surveys every 3 years. The major aim of the survey was to develop a 'literacy scale' for each of the three areas tested and to compare countries in terms of their average performance on these. The resulting continuous scales had distributions approximately Normal with a mean of 500 and standard deviation 100. Each scale was divided into six 'proficiency' categories each of which is exemplified in terms of responses to chosen sample test items. The scores were also analysed by background factors such as gender, and parental education and a multilevel analysis was also carried out to study variation between schools.

The three OECD literacy scales were constructed using model (18) with a single factor. Each scale used only those items designated as Reading, Mathematics or Science. Factor scores were computed for use in subsequent analyses. For these analyses a multiple imputation procedure was used as follows.

Each student has a factor score based on a linear predictor using their individual responses and the estimated model parameters. Under the model assumptions, these scores have an approximate Normal distribution, the accuracy of the approximation being a function of the number of item responses for an individual. Using an estimate of the standard error, multiple imputation is used; that is, a set of (typically 5) random draws from this estimated Normal distribution is made and these are then used for subsequent modelling (Rubin, 1996).

We shall refer to some of the limitations of this as a general procedure later, but for now note that a key feature is the use of a 1-dimensional factor model and we shall discuss below the incorporation of further dimensions.

For present purposes we have chosen the Mathematics test items for two countries, France and England. In total there are 31 Maths questions. In fact several questions are grouped in that they all relate to the same problem. For example, one problem described a pattern of trees planted as a set of squares of different sizes, and associated with this problem there were three separate questions. For a model such as (18) it is dubious whether for such questions the local independence assumption will hold, although this was assumed in the OECD analysis. A more satisfactory treatment would be to combine the three separate questions into an ordered scale, for example by forming an a priori suitably weighted combination of the responses, and treating this as an ordered categorical response as described above. For present purposes we have selected 15 items, each of which is a response to a different problem and dichotomised into correct/incorrect, treating part-correct answers as correct.

7. Estimation for the binary response factor model

The OECD analyses use the logit link function. The probit function generally produces similar results and has certain advantages in terms of computational convenience. One important advantage of the probit is that we can think of the response as a threshold from an underlying (unknown) continuous response, which is Normally distributed (Albert and Chib, 1993). We use the Gibbs sampling algorithm for Normally distributed responses described above and adapted for a probit model as follows.

Assume that we have a binary variable y_i collected for several individuals *i*, that is a threshold version of an (unknown) continuous normally distributed variable y_i^*. Now, if we knew the value of y_i^* then we could fit the standard Gibbs sampling algorithm for normal response models. So we add an extra step into the Gibbs sampling algorithm and generate y_i^* at each iteration from its conditional posterior distribution which is a truncated Normal distribution with mean (in the standard single-level probit model) $X\beta$ and variance 1. The truncation point is zero and if y_i is 0, y_i^* has to be negative and if y_i is 1, y_i^* has to be positive. This step is inserted into the existing algorithm for the Normal response factor model. It should be noted that this model can also be updated using Metropolis sampling but the Gibbs sampling algorithm is faster and produces less correlated chains. Consider the standard 2-level model

$$\begin{aligned} &Y = X\beta + ZU + e \\ &e \sim N(0,1) \end{aligned} \qquad (20)$$

Given current estimates of parameters and residuals, we have $Y \sim N(XB + ZU, 1)$ and for the probit model the observation of a positive value (>0) on the scale of Y corresponds to the observation of a 'success' on the probability scale and the observation of a negative (<0) value corresponds to a 'failure'. The probit function that determines the underlying chance of a correct response is the cumulative probability given by

$$\int_0^\infty \phi(t)dt, \quad \phi(t) \text{ is pdf of } N(X\beta + ZU, 1)$$

or equivalently

$$\int_{-(X\beta+ZU)}^{\infty} \phi(t)dt, \ \ \phi(t) \text{ is pdf of } N(0,1) \qquad (21)$$

Alternatively if we write the value of the *ij*-th response as

$$y_{ij} = (X\beta)_{ij} + (ZU)_{ij} + e_{ij}$$

a positive value occurs when $y_{ij} > 0$. We then have

$$\Pr(y_{ij} > 0) = \Pr(e_{ij} > -[(X\beta)_{ij} + (ZU)_{ij}]) \qquad (22)$$

which leads to (21)

Thus, given current values of β, U and the observation for a level 1 unit (0 or 1) we take a random draw e^*. If we observe a 1 then we draw from the truncated Normal distribution, $[-X^*, \infty]$, $X^* = (X\beta + ZU)$ and if we observe a 0 we sample from $[-\infty, -X^*]$. This is then applied to (20) to give a new value Y^*. This procedure is applied as an extra step in the factor analysis model, for example before step 1, with the remaining steps as before.

This approach is readily extended to the case of ordered categories, which can be applied to 'partial credit' models. We assume that there is an underlying Normally distributed response and that for *p*-category observed responses there are *p-1* thresholds. Assume a proportional odds model, where for the *s*-th cumulative probability we have (Goldstein, 2003)

$$\text{probit}(\gamma^{(s)}) = \alpha^{(s)} + (X\beta) \ + Z\ U$$

so that corresponding to (21) this gives

$$\int_{-(\alpha^{(s)} + X\beta + ZU)}^{\infty} \phi(t)dt$$

and sampling is conditional, as before, including the current values of the threshold parameters $\alpha^{(s)}$.

8. Results

The analyses reported below were carried out using MLwiN Beta version 1.2 (Browne, 2003, Rasbash et al., 2000). Table 1 shows a basic model in which a simple single level probit model is fitted for each item allowing for different country means.

(Table 1 here)

We see that, of the 10 statistically significant items, France does better on 4 (all free response items and worse on 6 (3 free response and 3 multiple choice items) than England. The interpretation of the probit function is that it predicts a value from an underlying standard Normal distribution with mean zero and standard deviation 1. This can be turned into a probability using the CDF of the standard Normal distribution. Thus, for example, the French students are, on average, 0.7 standard deviations ahead of the English for item 136Q01 (a free response Geometry item) but 0.7 standard deviations behind on item 161Q01 (a multiple choice Geometry item)

(Table 2 here)

We now fit a single factor at each level (the student and the school) with results in column A of Table 2. For convenience we present only the estimated factor loadings. At both levels we have a common factor with comparable loadings on each item, although at student level the multiple choice items tend to have smaller loadings. The next model fits a different mean for each item for France and England, namely

$$\text{probit}(\pi_{rijg}) = \beta_{0r} + \delta_g d_r + \beta_{1r}^{(1)}\theta_{1ij}^{(1)} + \beta_{1r}^{(2)}\theta_{1j}^{(2)} + u_{rj}$$
$$\theta_{1ij}^{(1)} \sim N(0,\sigma_{(1)}^2),\ \theta_{1j}^{(2)} \sim N(0,\sigma_{(2)}^2)$$
$$y_{rijg} \sim Bin(1,\pi_{rijg}),\ g = 1,2 \qquad (23)$$
$$\delta_g = \begin{cases} 0 \text{ if } g = 1 \\ 1 \text{ if } g = 2 \end{cases}$$

Note that in (23) g identifies country and takes values 1 for England and 2 for France, and we actually fit a global mean vector plus a difference term, d_r that captures the difference between French and English scores. The factor loadings are virtually unchanged. The means for the two countries, however, do differ somewhat for certain items. Thus, given the factor values, the French are somewhat further ahead than before on item 136Q01. This suggests that there may be different factor structures in the two countries, and we shall return to this below.

If we ignore the interaction between country and item it is then possible (but not otherwise) to use these models for purposes of comparing countries. There are two natural extensions where we allow the factor means to vary between countries but where the factor structures are the same in each country. Thus we can extend (23), for country *g,* as follows

$$\text{probit}(\pi_{rijg}) = \beta_{0rg} + \beta_{1r}^{(1)}\theta_{1ijg}^{(1)} + \beta_{1r}^{(2)}\theta_{1jg}^{(2)} + u_{rj},$$
$$\theta_{1ijg}^{(1)} \sim N(\mu_g^{(1)},\sigma_{(1)}^2),\ \theta_{1jg}^{(2)} \sim N(\mu_g^{(2)},\sigma_{(2)}^2) \qquad (24)$$
$$y_{rijg} \sim Bin(1,\pi_{rijg})$$

Typically we would be interested in modelling the same overall shift at each level (l) so that we have $\mu_g^{(l)} = \mu_g$. In this case for a single factor model (24) can be written in the alternative form

$$\text{probit}(\pi_{rijg}) = \beta_{0r} + \delta_g d(\beta_{1r}^{(1)} + \beta_{1r}^{(2)}) + \beta_{1r}^{(1)}\theta_{1ij}^{(1)} + \beta_{1r}^{(2)}\theta_{1j}^{(2)} + u_{rj},$$
$$\theta_{1ij}^{(1)} \sim N(0,\sigma_{(1)}^2),\ \theta_{1j}^{(2)} \sim N(0,\sigma_{(2)}^2) \qquad (25)$$
$$y_{rijg} \sim Bin(1,\pi_{rijg}),\ \ g = 1,2$$

Clearly we can extend such a model to other explanatory variables such as gender, in effect a structural equation model for the factor mean structures. We note that the OECD procedure for country and group comparisons will not in general produce the same inferences since the model that is fitted assumes no differences. Likewise, the OECD model assumes only a single (student) level with school level variation estimated in the second stage analysis. In the case of factor models the OECD approach to group comparisons leads to interpretational difficulties since the factor structure that is fitted, in the presence of real group differences under a model such as (23), is incorrect. We also note that for those with at least one valid mathematics item response (55%) the average number of mathematics items responded to by students is 12.6 with a range from 1 to 16, so that the Normal approximation implicit in the use of plausible values may not be very accurate for some of the students.

The OECD country comparisons for Mathematics show a small difference between the England and France and this is borne out by our own results, although we have used a reduced set of items.

An alternative formulation for country and group differences is to write (25) as

$$\text{probit}(\pi_{rij}) = \beta_{0r} + d\delta_g + \beta_{1r}^{(1)}\theta_{1ij}^{(1)} + \beta_{1r}^{(2)}\theta_{1j}^{(2)} + u_{rj}$$
$$\theta_{1ij}^{(1)} \sim N(0,\sigma_{(1)}^2),\ \theta_{1j}^{(2)} \sim MVN(0,\sigma_{(2)}^2) \qquad (26)$$
$$y_{rjk} \sim Bin(1,\pi_{rjk})$$

This model additionally constrains the item differences for each country to be constant. If we estimate the parameter d in (26) we obtain the value 0.02 with standard error 0.05 so that we would conclude that the French/English difference is small and non significant. In Table 2, however, we have shown considerable individual differences. Thus, if we had only fitted (26) representing a simple overall country effect, as in the PISA analysis, we would be missing potentially important differential (interaction) effects.

(Table 3 here)

The next model fits 2 orthogonal factors at student level and 1 at school level. In the present set of analyses we do not report fitting more than one factor at school level. In Table 3 the first factor at student level is again a general common factor, and the second factor tends to distinguish the free response from the multiple-choice items. We have also studied 3 factors at student level, but the results are not easy to interpret, perhaps unsurprisingly given only 15 binary response variables.

(Table 4 here)

We now fit separate factors for the two countries. Table 4 shows the results for a single factor at each level. We see that there are different patterns of loadings at both levels and those for France are much closer to the factor loadings estimated from the combined country dataset, perhaps unsurprisingly since there are almost twice as many French students in the combined sample. We have computed the factor scores for the English students from the combined and separate analyses and these show a high correlation (0.98). This reflects the fact that the factor score is effectively a weighted mean of the item responses, and the two sets of loadings are all positive and comparable in size. It is also inflated because the factor scores are 'shrunken' estimates with shrinkage a function of the number of items responded to. A simple comparison of the mean factor scores from the joint analysis with a single factor at each level gives a non-significant difference. Thus, while a joint analysis will lead to comparable rankings for individuals, as indeed will a simple scoring system using just the average percent correct (the correlation is 0.84), the interpretation of factor loadings in the context of group differences will not be the same.

At the school level for France the factor loadings are approximately proportional to those at school level for the combined analysis, but this is not the case for the UK, which has different orderings for the loadings. In the pooled analysis the comparison between student and school loadings is more like that for France.

9. Conclusions from the analysis

We have not attempted to study reasons for country differences in any detail in these analyses. Our intention has been to show how multilevel binary factor models can be specified with covariates and group differences. We show that for the purposes of comparing countries it is important to fit a model, which explicitly includes country effects. In the present case we show that, in the simple case where one general factor is fitted at each level, student and school, the factor structures are somewhat different for each country. Thus, a single 'pooled' set of factor loadings used for purposes of country comparisons leads to considerable difficulty in interpreting results, since, as in the present case, the pooled factors will be influenced by the weightings implicit in the numbers of students in each country. Where several countries are pooled as in the OECD PISA analyses, the factors are even more difficult to interpret as are resulting country differences. Furthermore, and perhaps more importantly, we have shown that, after fitting a single factor model there are still differences between countries in item response probabilities (Table 2). This implies that the choice of items to use will determine the factor loadings in that if we choose a majority of items that all load highly on a factor then that factor will tend to dominate the structure. If those items also happen to 'favour' a particular country then that country will tend to have higher factor scores, but this could only be ascertained by carrying out a multidimensional analysis.

10. Discussion

The issues that surround the specification and interpretation of single level factor and structural equation models are also present in our multilevel versions. Parameter identification has already been mentioned; with the ability to include prior distributions we can often treat identification problems with more flexibility. In the traditional model over-parameterisation requires setting one or more parameters or functions of parameters to known values. In our case we can obtain estimates by imposing informative prior distributions on each of the parameters which when combined with the data will provide the joint posterior distribution. An example is in the estimation of factor correlations where the assumption of a prior in the interval (0,1) can allow the joint posterior of all the parameters in an 'over-identified' model to be estimated.

Another potential advantage of our approach, common to all MCMC procedures, is that we can make exact inferences based upon the Markov chain values. This will be a particular advantage for small data sets where we may be unwilling to rely upon likelihood-based approximations.

Another issue is the boundary 'Heywood' case. We have observed such solutions occurring where sets of loading parameters tend towards zero or a correlation tends towards 1.0. A final important issue that only affects stochastic procedures is the problem of 'flipping states'. This means that there is not a unique solution even in a 1-factor problem as the loadings and factor values may all flip their sign to give an equivalent solution. When the number of factors increases there are greater problems as factors may swap over as the chains progress. This means that identifiability is an important consideration when using stochastic techniques.

We can extend the models considered here to mixtures of binary, ordered and continuous responses. We have separately discussed all three types of responses. They are linked via the threshold probit model so that at level 1 we have a set of independent Normal variables (uniquenesses), each one arising from a continuous response, a binary response or an ordered response. At higher levels the random effects are assumed to have a multivariate Normal distribution and the MCMC estimation proceeds in a straightforward fashion.

Such an example might arise in a health application where individuals respond to a health questionnaire at the same time as a set of continuous measurements of health status are made. It might also arise in an educational examination where, for example, some responses are multiple-choice binary questions and some are free responses marked on a continuous scale. Another important application is to questionnaires that contain mixtures of ordered rating scales and binary responses.

A major drawback of current implementations of binary factor (item response) models that attempt to account for multilevel data structures, is that fit a multilevel model in two stages: first by estimating a single level model and then fitting a multilevel model using the estimated factor scores, typically using multiple imputation via plausible values. Such a procedure does not allow the exploration of any factor structure at higher levels. We have shown in our example that this may be important, especially when comparing groups or countries.

Finally, we note that all of our models can be extended straightforwardly to more complex data structures involving cross-classifications and multiple membership structures (Browne et al., 2001).

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Tables

Table 1. **Separate country analyses with probit response model for each item. Columns show UK mean and French – English difference between means. Standard errors in brackets. 10,000 MCMC iterations with default priors. The type of item is shown by each item name (MC = multiple choice; FR = free response)**

Item	**England**	***France-England***
Student level		
33q01 (MC)	0.80	-0.06 (0.05)
34q01 (FR)	-0.25	0.03 (0.06)
37q01 (FR)	0.65	-0.11 (0.07)
124q01 (FR)	0.01	-0.18 (0.07)
136q01 (FR)	-0.23	0.69 (0.05)
144q01 (FR)	0.16	0.40 (0.05)
145q01 (FR)	0.65	-0.13 (0.06)
150q01 (FR)	0.78	-0.35 (0.06)
155q01 (FR)	0.54	0.27 (0.06)
159q01 (MC)	0.89	-0.24 (0.06)
161q01 (MC)	0.96	-0.70 (0.06)
179q01 (FR)	-0.11	0.64 (0.06)
192q01 (MC)	-0.28	0.07 (0.06)
266q01 (MC)	-0.75	-0.26 (0.06)
273q01 (MC)	-0.04	0.03 (0.06)

Table 2. Factor loadings with a single factor at each level. Analysis A ignores country differences; analysis B fits model (23) and shows the loadings together with French – English difference. Factor variances are set equal to 1.			
	A	***B***	
		Loadings	***France -England***
Item			
Student level			
33q01	0.46 (0.04)	0.46 (0.04)	-0.08 (0.04)
34q01	0.70 (0.04)	0.71 (0.05)	0.01 (0.07)
37q01	0.96 (0.09)	0.92 (0.07)	-0.16 (0.10)
124q01	0.69 (0.07)	0.72 (0.07)	-0.20 (0.10)
136q01	0.69 (0.05)	0.70 (0.06)	0.96 (0.08)
144q01	0.55 (0.05)	0.54 (0.05)	0.46 (0.07)
145q01	0.63 (0.05)	0.62 (0.05)	-0.21 (0.08)
150q01	0.59 (0.05)	0.59 (0.04)	-0.41 (0.07)
155q01	0.51 (0.05)	0.52 (0.04)	0.33 (0.07)
159q01	0.46 (0.04)	0.47 (0.05)	-0.31 (0.07)
161q01	0.30 (0.04)	0.33 (0.04)	-0.78 (0.07)
179q01	0.54 (0.06)	0.52 (0.06)	0.79 (0.08)
192q01	0.68 (0.04)	0.68 (0.05)	0.09 (0.07)
266q01	0.36 (0.05)	0.38 (0.05)	-0.28 (0.07)
273q01	0.47 (0.04)	0.46 (0.05)	0.03 (0.06)
School level			
33q01	0.26 (0.03)	0.26 (0.03)	
34q01	0.39 (0.03)	0.39 (0.03)	
37q01	0.77 (0.06)	0.74 (0.06)	
124q01	0.71 (0.05)	0.71 (0.05)	
136q01	0.49 (0.04)	0.54 (0.04)	
144q01	0.31 (0.04)	0.31 (0.03)	
145q01	0.47 (0.04)	0.48 (0.04)	
150q01	0.41 (0.04)	0.42 (0.04)	
155q01	0.29 (0.04)	0.31 (0.03)	
159q01	0.35 (0.04)	0.36 (0.04)	
161q01	0.23 (0.04)	0.24 (0.03)	
179q01	0.42 (0.05)	0.46 (0.04)	
192q01	0.43 (0.03)	0.43 (0.04)	
266q01	0.33 (0.04)	0.33 (0.04)	
273q01	0.35 (0.03)	0.35 (0.03)	

Table 3. **Loadings for two orthogonal factors at level 1 and one factor at level 2. First loading of factor 2 constrained to zero. Variances constrained to one.**		
Item	**Factor 1**	***Factor 2***
Student level		
33q01	0.51 (0.06)	0
34q01	0.67 (0.05)	0.22 (0.09)
37q01	0.81 (0.10)	0.42 (0.14)
124q01	0.56 (0.11)	0.80 (0.21)
136q01	0.60 (0.09)	0.47 (0.12)
144q01	0.58 (0.10)	0.08 (0.10)
145q01	0.57 (0.06)	0.19 (0.12)
150q01	0.72 (0.10)	-0.07 (0.18)
155q01	0.44 (0.06)	0.28 (0.10)
159q01	0.50 (0.06)	-0.04 (0.12)
161q01	0.43 (0.07)	-0.27 (0.14)
179q01	0.46 (0.08)	0.46 (0.17)
192q01	0.62 (0.06)	0.28 (0.10)
266q01	0.41 (0.06)	-0.10 (0.09)
273q01	0.42 (0.06)	0.21 (0.12)
School level		
33q01	0.27 (0.03)	
34q01	0.39 (0.04)	
37q01	0.76 (0.06)	
124q01	0.82 (0.10)	
136q01	0.52 (0.05)	
144q01	0.32 (0.04)	
145q01	0.47 (0.04)	
150q01	0.45 (0.05)	
155q01	0.31 (0.04)	
159q01	0.36 (0.04)	
161q01	0.25 (0.04)	
179q01	0.46 (0.05)	
192q01	0.44 (0.04)	
266q01	0.34 (0.04)	
273q01	0.36 (0.03)	

***Table 4.* Loadings for single factor models separately for each country.**		
Item	**England**	***France***
Student level		
33q01	0.49 (0.08)	0.46 (0.04)
34q01	0.56 (0.10)	0.75 (0.05)
37q01	0.69 (0.12)	1.09 (0.11)
124q01	0.50 (0.10)	0.82 (0.09)
136q01	0.75 (0.14)	0.71 (0.06)
144q01	0.48 (0.09)	0.58 (0.06)
145q01	0.38 (0.09)	0.68 (0.05)
150q01	0.50 (0.11)	0.62 (0.06
155q01	0.31 (0.08)	0.59 (0.05)
159q01	0.34 (0.09)	0.51 (0.06)
161q01	0.32 (0.09)	0.33 (0.05)
179q01	0.34 (0.09)	0.62 (0.07)
192q01	0.75 (0.14)	0.68 (0.05)
266q01	0.33 (0.09)	0.41 (0.06)
273q01	0.44 (0.09)	0.48 (0.05)
School level		
33q01	0.36 (0.06)	0.22 (0.04)
34q01	0.46 (0.07)	0.36 (0.04)
37q01	0.73 (0.09)	0.79 (0.08)
124q01	0.80 (0.10)	0.68 (0.06)
136q01	0.75 (0.08)	0.44 (0.04)
144q01	0.41 (0.06)	0.25 (0.04)
145q01	0.72 (0.08)	0.39 (0.04)
150q01	0.44 (0.06)	0.43 (0.04)
155q01	0.23 (0.06)	0.35 (0.04)
159q01	0.47 (0.07)	0.31 (0.04)
161q01	0.27 (0.06)	0.24 (0.04)
179q01	0.50 (0.07)	0.45 (0.05)
192q01	0.42 (0.08)	0.44 (0.04)
266q01	0.38 (0.07)	0.30 (0.05)
273q01	0.41 (0.06)	0.32 (0.04)