Reflecting on truth in a partial setting

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Overview

- The study of reflection principles are important in the arithmetical setting.
- Also for theories of truth the investigation of reflection principles is important and fruitful.
- What about reflection principles in a partial setting?
- What about the connection between reflection and PKF?
1 Background
- Axiomatizing Kripke
- \( \mathbb{N} \)-Categoricity
- Infinitary proof systems

2 Reflection
- From the \( \omega \)-rule to reflection
- From Tarski biconditionals to KF

3 Reflecting on truth in a partial setting
- Partial logic
- Recovering PKF
- Induction
Kripke models

- Kripke: Fixed-point construction for different evaluation schemes $e$.
- Monotone operators $\Gamma_e$.
- Fixed-points $\Gamma_e(S) = S$ for $S \subseteq \mathbb{N}$.
- Focus: strong Kleene, $e = sk$.
- The minimal fixed-point for strong Kleene $I_{sk}$.
Axiomatizing Kripke

Axiomatizations:

- **KF (Feferman)**
  The problem of external and internal logic.

- **IKF (Reinhardt)** ($\{ A \in \mathcal{L}_T \mid \text{KF} \vdash T(\neg \neg A) \}$)
  The problem of natural axiomatization.

- **PKF (Halbach/Horsten)**

In what sense are these axiomatizations and which one is preferable?
**N-Categoricity**

Suggestion: N-categoricity.

Fix the interpretation of the arithmetical part with the standard model \( \mathcal{N} \). \( \Sigma \) is N-categorical for a set of models \( M \) iff

\[
(\mathcal{N}, S) \models \Sigma \iff S \in M
\]

For the minimal fixed-point:

\[
(\mathcal{N}, S) \models \Sigma \iff S = I_{sk}
\]

For arbitrary fixed-points:

\[
(\mathcal{N}, S) \models \Sigma \iff S = \Gamma_{sk}(S)
\]
The minimal fixed-point is $\Pi^1_1$-complete (Kripke, Burgess).

There is no $\mathbb{N}$-categorical axiomatization of the minimal fixed-point.

KF is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.

(Feferman)

TFB is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.

(Leigh)

IKF is not $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.

Conclusion: KF is at best an axiomatization of arbitrary fixed-points and $\mathbb{N}$-categoricity cannot be the only criterion.
N-Categoricity and partiality

- The set of derivable sequents of PKF is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.
- The set of theorems of PKF, i.e. sequents of the form $\Rightarrow A$, is not $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.
- The set of truth sequents $T(\neg A \neg) \Rightarrow A$, $A \Rightarrow T(\neg A \neg)$ is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.
Infinitary proof systems

Infinitary proof systems allow for characterizations of the minimal fixed-points.

- Cantini has an infinitary proof system (sequent system with $\omega$-rule) characterizing the minimal fixed-point of supervaluation.
- Welch gametheoretic characterization.
- Meadows infinitary tableaux.
Infinitary proof system for strong Kleene

Example $SK_\infty$ a Tait system: Initial sequents

$\Rightarrow A$ (for true atomic arithmetical sentences)

$\Rightarrow A \quad \Rightarrow A(\Gamma, T(\Gamma A^-)) \quad \Rightarrow \neg A \quad \Rightarrow \Gamma, \neg T(\Gamma A^-)$

$\omega$-rule

... $A(n)$ ...

$\forall x A(x)$ ...

(for all $n \in \mathbb{N}$)

Then

$SK_\infty \vdash A \iff \#A \in I_{sk}$
Embeddings into infinitary proof systems

Similar to the Gentzen-Schütte method we can look at embeddings into the infinitary proof systems.

- KF cannot be directly embedded.
- An embedding of the theorems of PKF into $\text{SK}_\infty$ is possible
  - if $\text{PKF} \vdash \Rightarrow A$, then $\#A \in \Gamma_\omega$ (Cantini, Halbach/Horsten).
  - for the language of truth we only have transfinite induction up to $\omega^\omega$ in PKF.
- IKF is contained in $\text{I}_{sk}$
  - if $\text{IKF} \vdash A$, then $\#A \in \Gamma_\epsilon_0$ (Cantini).
  - for the language of truth we have transfinite induction up to $\epsilon_0$ in KF.
Uniform reflection as a finitary $\omega$-rule

\[ (RFN^R_\Sigma) \quad \frac{\forall x \Pr_\Sigma(\Gamma \vdash A^x)}{\forall x A(x)} \]

\[ (RFN_\Sigma) \quad \forall x (\Pr_\Sigma(\Gamma \vdash A^x) \rightarrow A(x)). \]

- Hilbert 1931.
- Shoenfield constructivized version of the $\omega$-rule.
- Feferman 1962 showed the equivalence.
The strength of uniform reflection

For an axiomatizable theory \( \Sigma \) we use \( R(\Sigma) := EA_T + RFN_\Sigma \).

- \( TB_0 \) is \( EA_T + \) Tarski biconditionals for sentences of \( L_A \).
- \( UTB_0 \) is \( EA_T + \) uniform Tarski biconditionals for formulas of \( L_A \).
- \( TFB_0 \) is \( EA_T + \) truth and falsity biconditionals for sentences of \( L_P \), i.e. the language of we get by adding \( F \) as the dual for \( T \) and allow only positive occurrences of \( T \) and \( F \).

\[
T(\neg A) \iff A \& F(\neg A) \iff \neg A
\]

- \( UTFB_0 \) is \( EA_T + \) uniform truth and falsity biconditionals for formulas of \( L_P \).
Truth and Reflection

- Reflecting on Tarski biconditionals gives uniform Tarski biconditionals.

**Lemma (Horsten, Leigh)**

\[ \text{UTB}_0 \subseteq R(\text{TB}_0). \]

- Reflecting on typefree truth and falsity biconditionals gives uniform typefree truth and falsity biconditionals.

**Lemma (Horsten, Leigh)**

\[ \text{UTFB}_0 \subseteq R(\text{TFB}_0). \]
Truth and Reflection

- Reflecting on uniform Tarski biconditionals gives the compositional axioms.

**Lemma (Halbach)**

\[ CT_0 \subseteq R(UTB_0). \]

- Reflecting on uniform truth and falsity biconditionals gives the compositional axioms of KF.

**Lemma (Horsten, Leigh)**

\[ KF \subseteq R(UTFB_0). \]
Partial logic

- The logic is four valued.
- Gaps and gluts.
- Logical consequence for sequents:
  - Truth preservation
  - Falsity antipreservation
Basic

For negation we have contraposition

\[
\Gamma \Rightarrow \Delta \\
\neg \Delta \Rightarrow \neg \Gamma
\]

but not

\[
\begin{align*}
A, \Gamma \Rightarrow \Delta \\
\Gamma \Rightarrow \Delta, \neg A
\end{align*}
\]

\[
\begin{align*}
\Gamma \Rightarrow \Delta, A \\
\neg A, \Gamma \Rightarrow \Delta
\end{align*}
\]

We assume as background an arithmetical theory BASIC formulated in $\mathcal{L}_T$: EA$_T$ formulated in a sequent version of partial logic along the lines of Halbach 2014.
Minimal truth $TS_0$

$TS_0$ is obtained by extending BASIC with the initial sequents

\[ T1 \quad T(\lnot A) \Rightarrow A \]
\[ T2 \quad A \Rightarrow T(\lnot A) \]

- Simplicity.
- No need for restriction of the language.
Reflection as a rule

Assume some coding of finite sets of formulas $[\Gamma]$, then $[\Gamma \dot{x}]$ denotes the result of substituting in $\Gamma$ the $x$-th numeral for $x$.

$$[\Gamma \dot{x}] \Rightarrow [\Delta \dot{x}]$$

denotes the sequent $\Gamma(x) \Rightarrow \Delta(x)$ with the possible free variable $x$ and the dots indicate as usual the use of the sub and num function.

Let $\Sigma$ be an axiomatizable theory, then $R(\Sigma) = EA_T + RFN^R_\Sigma$.

$$\begin{align*}
(RFN^R_\Sigma) & \quad \text{Pr}_\Sigma([\Gamma \dot{x}] \Rightarrow [\Delta \dot{x}]) \\
\Gamma(x) & \Rightarrow \Delta(x)
\end{align*}$$
From $\text{TS}_0$ to $\text{UTS}_0$

$\text{R} (\text{TS}_0) \vdash$

(i) $A(x) \Rightarrow T (\neg A x \neg)$;

(ii) $T (\neg A x \neg) \Rightarrow A(x)$.

Argument: For all formulas $A(x)$ and for all $n \in \mathbb{N}$:

$\text{TS}_0 \vdash A(n) \Rightarrow T (\neg A(n) \neg)$.

As this is uniform we get in the formalization

$\text{EA}_{T} \vdash \text{Pr}_{\text{TS}_0} ([A x] \Rightarrow [T (\neg A \neg) x])$.

With reflection we get

$\text{R} (\text{TS}_0) \vdash A(x) \Rightarrow T (\neg A x \neg)$.
Regaining compositional sequents I

\[ R(TS_0) \vdash \]

(i) \( \text{sent}(x), \text{sent}(y), T(x \land y) \Rightarrow T(x) \land T(y); \)

(ii) \( \text{sent}(x), \text{sent}(y), T(x) \land T(y) \Rightarrow T(x \land y); \)

(iii) \( \text{sent}(x), \text{sent}(y), T(x \lor y) \Rightarrow T(x) \lor T(y); \)

(iv) \( \text{sent}(x), \text{sent}(y), T(x) \lor T(y) \Rightarrow T(x \lor y); \)

(v) \( \text{sent}(x), \neg T(x) \Rightarrow T(\neg x); \)

(vi) \( \text{sent}(x), T(\neg x) \Rightarrow \neg T(x). \)
Regaining compositional sequents II

\[
\text{R}(\text{UTS}_0) \vdash \\
\begin{align*}
(i) & \text{ sent}(\forall xy), \forall x T(y \dot{x}) \Rightarrow T(\forall xy); \\
(ii) & \text{ sent}(\forall xy), T(\forall xy) \Rightarrow \forall x T(y \dot{x}); \\
(iii) & \text{ sent}(\exists xy), \exists x T(y \dot{x}) \Rightarrow T(\exists xy); \\
(iv) & \text{ sent}(\exists xy), T(\exists xy) \Rightarrow \exists x T(y \dot{x}).
\end{align*}
\]
Regaining compositional sequents III

\[ R(UTS_0) \vdash \]

(i) \( ct(x), T(val(x)) \Rightarrow T(\neg x) \);
(ii) \( ct(x), T(\neg x) \Rightarrow T(val(x)) \);
(iii) \( ct(x), ct(y), val(x) = val(y) \Rightarrow T(x \equiv y) \);
(iv) \( ct(x), ct(y), T(x \equiv y) \Rightarrow val(x) = val(y) \).

Observation

\( \text{PKF}_0 \subseteq R(UTS_0) \subseteq R(R(TS_0)) \)
Induction in classical arithmetic

Theorem (Kreisel and Lévy)

$R(\text{EA}) = \text{PA}$. 

Argument for $\supseteq$:

For a formula $A$ with one free variable let $B(x)$ be $A(0) \land \forall x (A(x) \rightarrow A(x+1)) \rightarrow A(x)$. Then we can argue in EA by external induction that for all $k$, $\text{EA} \vdash B(k)$. Since the size of the proofs can be bound by an elementary function we can formalize the induction in EA. So we get $\text{EA} \vdash \text{Pr}_{\text{EA}}(\neg B\xi)$ and with reflection $B(x)$.

Similarly we get $R(\text{EA}_T) = \text{PA}_T$.  

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Reflecting on truth in a partial setting  
Bristol-München 24 / 36
Induction for $\mathcal{L}_T$ (partial)

Instead of using the (schema) of induction, the following rule is adopted:

\[
\frac{A(x), \Gamma \Rightarrow \Delta, A(x+1)}{A(0), \Gamma \Rightarrow \Delta, A(t)} \quad \text{(Ind)}
\]

In $R(UTS_0)$ we get induction for all formulas of $\mathcal{L}_T$ and so

**Observation**

$PKF \subseteq R(UTS_0) \subset R(R(TS_0))$. 
Transfinite induction

For a fixed ordinal representation, for example with the Cantor normal form for ordinals $< \varepsilon_0$ we define:

**Definition**

Let $A$ be a formula with one free variable

- $Prog(A) := \forall \alpha < \beta A(\alpha) \rightarrow A(\beta)$.
- $TI(A, \beta) := Prog(A) \rightarrow \forall \alpha < \beta A(\alpha)$.
- $TI_L(< \alpha) := \{ TI(A, \beta) \mid A \in \mathcal{L} \& \beta < \alpha \}$.
Transfinite induction for a language with truth

Lemma

Reflecting on $EA_T$ gives $TI_{L_T}(\langle \varepsilon_0 \rangle)$.

Argument: Similar to PA proves transfinite induction up to $\varepsilon_0$. For a formula $A(x)$ define $A'(x)$ to be

$$\forall \beta (\forall \alpha < \beta A(\alpha) \rightarrow \forall \alpha < \beta + \omega^{\alpha} A(\alpha))$$

Then we show

$$Prog(A) \rightarrow Prog(A').$$

With this

$$TI(A, \alpha) \Rightarrow TI(A, \omega^{\alpha}),$$

and finally

$$TI_{L_T}(\langle \varepsilon_0 \rangle).$$
$\text{TI}_{\mathcal{L}_T}$ in a partial setting

$\text{Prog}(A) := \forall \alpha < \beta A(\alpha) \Rightarrow A(\beta)$

$\text{TIR}(A, \beta) \quad \frac{\text{Prog}(A)}{\Rightarrow \forall \alpha < \beta A(\alpha)}$

$\text{TIR}_{\mathcal{L}_T}(< \alpha)$ is the closure under the rules $\text{TIR}(A, \beta)$ for all $A \in \mathcal{L}_T$ and for all $\beta < \alpha$. 
TIR_{\mathcal{L}_T}(< \epsilon_0) in \text{R(UTS)}? \\

Basic proof strategy: Show \\

\[
\frac{\text{Prog}(A)}{\text{Prog}(A')}
\]

then closure under TIR(A, \beta) implies closure under TIR(A, \omega^\beta) for all A \in \mathcal{L}_T.
Problems for the direct argument

We run into problems if we try to show that

\[
\frac{\text{Prog}(A)}{\text{Prog}(A')}
\]

Remember that \(A'(x)\) is \(\forall \beta (\forall \alpha < \beta A(\alpha) \rightarrow \forall \alpha < \beta + \omega^x A(\alpha))\).

In our partial setting we do not have in general

\[
\Rightarrow A \quad \Rightarrow A \rightarrow B \quad \Rightarrow B
\]
Idea

Idea (Carlo): circumvent the MP argument step.
In UTS we can prove (by external induction) for all $n$ that

$$
	ext{Prog}(A) \quad \frac{\forall \alpha < \beta A(\alpha)}{\forall \alpha < \beta + \omega^n A(\alpha)}
$$

Problem: How to use this fact?
Reflection on rules

Solution: Strengthening of reflection. Assume that \( \Sigma \) allows for the following derivation

\[
\Gamma \Rightarrow \Delta \\
\Theta \Rightarrow \Lambda
\]

Then a reflection on \( \Sigma \) should also include this fact

\[
(R^*) \quad \text{Pr}_\Sigma([\Gamma x] \Rightarrow [\Delta x], [\Theta x] \Rightarrow [\Lambda x]) \quad \Gamma(x) \Rightarrow \Delta(x) \\
\Theta(x) \Rightarrow \Lambda(x)
\]
TIR_{\mathcal{L}_T}(< \varepsilon_0) in R^*(UTS)

Now we can formalize the external induction to get

$$\Pr_{UTS}([\text{Prog}(A)], [\forall \alpha < \beta A(\alpha)] \Rightarrow [\forall \alpha < \beta + \omega^x A(\alpha)\dot{x}])$$

and with reflection we have

$$\frac{\text{Prog}(A)}{\forall \alpha < \beta A(\alpha) \Rightarrow \forall \alpha < \beta + \omega^x A(\alpha)}$$

Setting $\beta = 0$ we can then argue for $TIR_{\mathcal{L}_T}(< \varepsilon_0)$. 
Open questions

- In classic theories we have a close connection between reflection and induction.
- Is it as close in partial logic?
- Is reflection able to close the (proof theoretic) gap between PKF and IKF?
Concluding remarks

- Theories of truth built on reflection principles are very well motivated.
- Reflection on simple truth sequents allows us to gain the compositional axioms of PKF.
- Reflection and induction are closely connected also in the partial setting.
- Reflection gives full induction.
- Reflection gives $\text{TIR}_{\mathcal{L}_T}(\prec \epsilon_0)$. 
Thank you!