What is the biggest possible gap between quantum and classical computing?

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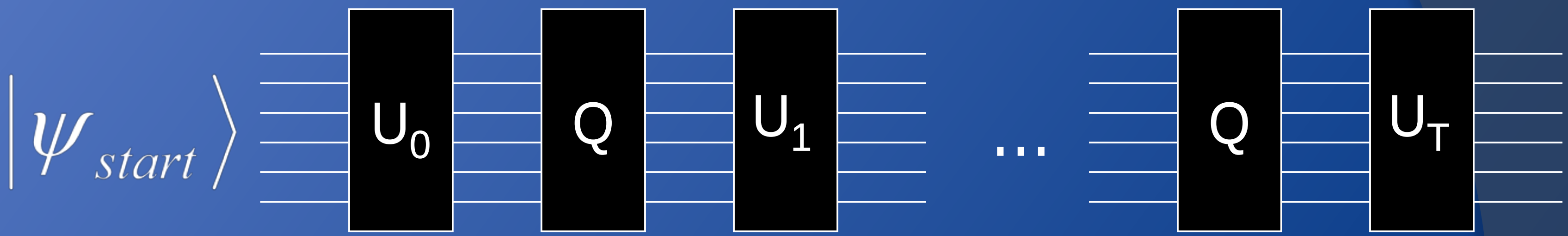
Query model

- Function $f(x_1, ..., x_N)$, $x_i \in \{0,1\}$.
- x_i given by a black box:

Complexity = number of queries

Quantum query model



- Q – queries: $$\sum_i a_i |i\rangle \rightarrow \sum_i a_i (-1)^{x_i} |i\rangle$$
- U_0, U_1, …, U_T – independent of x_1, …, x_N.

Reasons to study query model

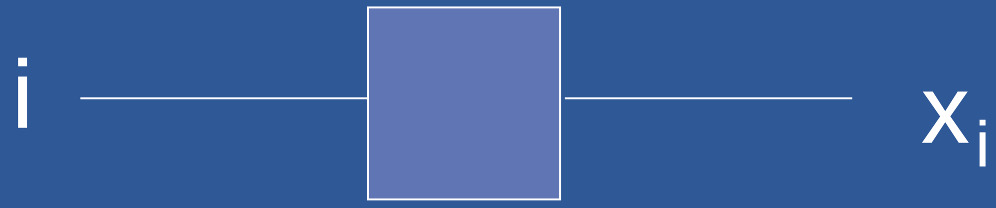
- Encompasses many quantum algorithms (Grover's search, quantum part of factoring, etc.).
- Provable quantum-vs-classical gaps.

1 query quantumly

How many queries classically?

Period finding

$x_1, x_2, \ldots, x_N$ - periodic



Find period r

1 query quantumly

Quantum part of Shor’s factoring algorithm

How many queries classically?

$$\underbrace{\underbrace{1, 2, 3, \ldots, 7}_{r}, \underbrace{1, 2, 3, \ldots, 7}_{r}, \ldots}_{N}$$

- Quantum algorithm works if $N \geq r^2$.
- T classical queries – can test T^2 possible periods.

$c\sqrt[4]{N}$ queries classically

Our result [Aaronson, A]

- Task that requires 1 query quantumly, $\Theta(\sqrt{N})$ classically.
- Method for simulating any 1 query quantum algorithm by $O(\sqrt{N})$ query probabilistic algorithm.

Fourier checking/Forrelation

Forrelation

- Input: $(x_1, \ldots, x_N, y_1, \ldots, y_N) \in \{-1, 1\}^{2N}$.
- Are vectors

$$\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_N \end{pmatrix} \qquad F_N \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_N \end{pmatrix}$$

well correlated one with another?

- F_N – Fourier transform over Z_N.

More precisely...

- Is the inner product

$$\left(\vec{x},\ F\vec{y}\right)=\frac{1}{N}\sum_{i,j}F_{i,j}x_i y_j$$

at least 3/5 or at most 1/100?

Quantum algorithm

1. Generate states

$$\left|\Psi_x\right\rangle = \sum_{i=1}^{N} x_i \left|i\right\rangle, \quad \left|\Psi_y\right\rangle = \sum_{i=1}^{N} y_i \left|i\right\rangle$$

   in parallel (1 query).
2. Apply F_N to 2^{nd} state.
3. Test if states equal (SWAP test).

Classical lower bound

- Theorem Any classical algorithm for FORRELATION uses

$$\Omega\left(\frac{\sqrt{N}}{\log N}\right)$$

queries.

REAL FORRELATION

- Real-valued vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_N \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_N \end{pmatrix}$$

- Distinguish between
  - $\vec{x}, \vec{y}$ random (x_i's - Gaussian);
  - $\vec{x}$ random, $\vec{y} = F_N \vec{x}$.

Reduction

T query algorithm for FORRELATION

↓

T query algorithm for REAL FORRELATION

- Proof idea: achieve $x_i \in \{-1, 1\}$ by replacing $x_i \rightarrow \mathrm{sgn}(x_i)$.

Lower bound

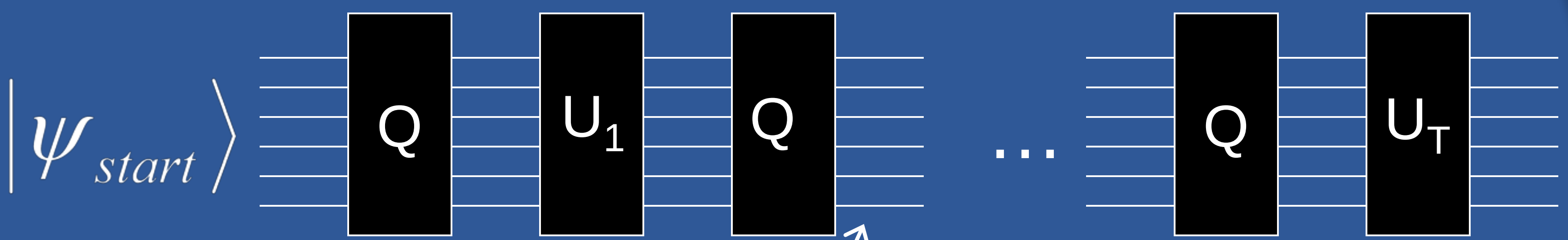
- Claim Solving REAL FORRELATION on most instances requires $\Omega\left(\frac{\sqrt{N}}{\log N}\right)$ queries.
- Intuition: if $\vec{y} = F_N \vec{x}$, correlations between x_i's and y_j's - weak.
- $o(\sqrt{N})$ values x_i and y_j look like uncorrelated random variables.

Simulating 1 query quantum algorithms

Simulation

- Theorem Any 1 query quantum algorithm computing $f(x_1, \ldots, x_N)$ can be simulated probabilistically using $O(\sqrt{N})$ queries.

Analyzing query algorithms



$$\alpha_{1,1}|1,1\rangle + \alpha_{1,2}|1, 2\rangle + \ldots + \alpha_{N, M}|N, M\rangle$$

$\alpha_{1,1}$ is actually $\alpha_{1,1}(x_1, \ldots, x_N)$

Polynomials method

- Lemma [Beals et al., 1998] If

$$\sum_{i,j} \alpha_{i,j}(x_1, \ldots, x_N)\left|i, j\right\rangle$$

is a state after k queries, then $\alpha_{i,j}(x_1, \ldots, x_N)$ are polynomials in $x_1, \ldots, x_N$ of degree $\le k$.

Measurement:
(i, j) w. probability $\left|\alpha_{i,j}(x_1, \ldots, x_N)\right|^2$

Polynomial of degree $\le 2k$

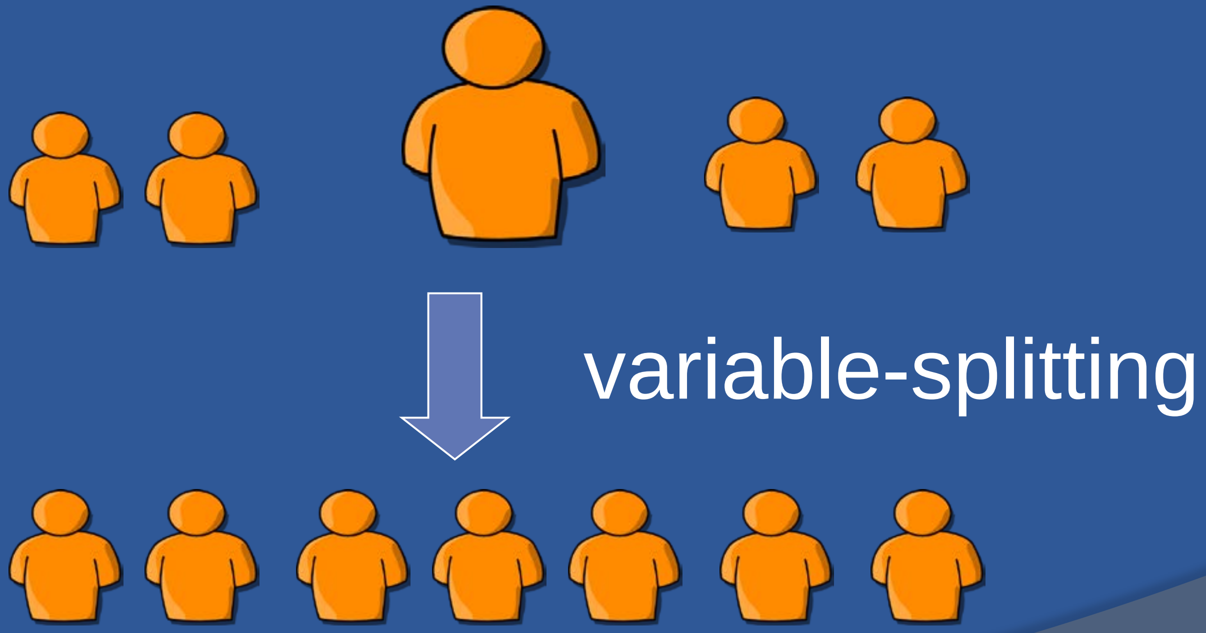
Our task

- $\Pr[A \text{ outputs } 1] = p(x_1, ..., x_N)$, deg $p = 2$.
- $0 \leq p(x_1, ..., x_N) \leq 1$.
- Task: estimate $p(x_1, ..., x_N)$ within precision ε.

Solution: random sampling

Pre-processing

- Problem: some x_i's in $p(x_1, ..., x_N)$ may be more influential than others.



Sampling 1

$$p(x_1, x_2, \ldots, x_N) = \sum_{i,j} a_{i,j} x_i x_j$$

- Claim If we sample N out of N^2 terms $Y_{i,j}=a_{i,j}x_ix_j$, then

$$\sum_{i,j-sampled} Y_{i,j}$$ - good estimate

Problem: requires sampling N variables x_i.

Sampling 2

$$p(x_1, x_2, \ldots, x_N) = \sum_{i,j} a_{i,j} x_i x_j$$

$\boxed{\text{Sampling N terms } Y_{i,j}=a_{i,j}x_ix_j}$

|||

$\boxed{\text{Sampling } \sqrt{N} \text{ variables } x_i}$ $\sqrt{N} \bullet \sqrt{N} = N$

Extension to k queries

- Theorem Any k query quantum algorithm can be simulated probabilistically with $O(N^{1-1/2k})$ queries.
- Proof Describe algorithm by polynomial of degree 2k, use random sampling.
- Question: Is this optimal?

K-fold forrelation

- Forrelation: given black box functions f(x) and g(y), estimate

$$\sum_{x,y} F_{x,y} f(x) g(y)$$

- K-fold forrelation: given $f_1(x)$, ..., $f_k(x)$, estimate

$$\sum_{x_1,\ldots,x_k} f_1(x_1) F_{x_1,x_2} f_2(x_2) F_{x_2,x_3} \ldots f_k(x_k)$$

k-query quantum algorithm

1. Generate $\sum_x \frac{1}{\sqrt{N}} |x\rangle$
2. Apply black box for $f_1(x)$;
3. Apply QFT;
4. Apply black box for $f_2(x)$;
5. ....

Creates amplitude equal to

$$\sum_{x_1,\ldots,x_k} f_1(x_1) F_{x_1,x_2} f_2(x_2) F_{x_2,x_3} \ldots f_k(x_k)$$

More results

- Theorem k-fold forrelation can be solved with $\lceil k/2 \rceil$ quantum queries.
- Conjecture k-fold forrelation requires $\Omega(N^{1-1/k})$ queries classically.
- $\Omega(N^{1-1/k})$ queries = estimating the sum by classical sampling.

BQP-completeness

- Let $k = \text{poly}(n)$ and $f_1(x), \ldots, f_k(x)$ - poly-size quantum circuits.
- Theorem k-fold forrelation is BQP-complete.
- Captures the power of BQP!
- No Jones polynomial or other advanced notions!

BQP-completeness proof

- Need to show:

  poly-size quantum circuits $\Rightarrow$ k-fold forrelation.
- Hadamard + sign (cc-Z) – universal.
- Transformation:
  - Sign gates $\Rightarrow$ $f_1(x)$, $f_2(x)$, ..., $f_k(x)$;
  - Hadamard $\Rightarrow$ Fourier transform;

1 quantum query algorithms for sampling problems

Fourier sampling

- Black box for f(x), x$\in$\{0,1\}^{N}.
- Probability distribution $P[y] = \left(\hat{f}(y)\right)^2$,

$$\hat{f}(y) = \frac{1}{\sqrt{2^N}} \sum_x F_{x,y} f(x).$$

- Task: sample from this distribution.

Quantum algorithm

1. Use 1 query to generate

$$|\Psi\rangle = \frac{1}{\sqrt{2^N}} \sum_x f(x) |x\rangle,$$

2. Apply QFT to obtain

$$|\Psi'\rangle = \sum_y \hat{f}(y) |y\rangle,$$

$$\hat{f}(y) = \frac{1}{\sqrt{2^N}} \sum_x F_{x,y} f(x).$$

Classical lower bound

- Theorem Fourier sampling requires $\Omega(N/\log N)$ queries, even for approximate sampling

Summary

- 1 quantum query = $\Theta(\sqrt{N})$ classical queries.
- k quantum queries can be simulated with $O(N^{1-1/2k})$ classical queries.
- Sampling: at least $\Omega(N/\log N)$ classical queries to simulate 1 quantum query.

Open problem 1

- Does k-fold FORRELATION require $\Omega(N^{1-1/2k})$ queries classically?
- Plausible but looks quite difficult matematically.

Open problem 2

- Best quantum-classical gaps:
  - 1 quantum query - $\Omega(\sqrt{N})$ classical queries;
  - 2 quantum queries - $\Omega(\sqrt{N})$ classical queries;
  - ...
  - log N quantum queries - $\Omega\left(\sqrt{N \log N}\right)$ classical queries.

Any problem that requires O(log N) queries quantumly, $\Omega(N^c)$, c>1/2 classically?

Open problem 3

- What else is FORRELATION/k-fold FORRELATION useful for?