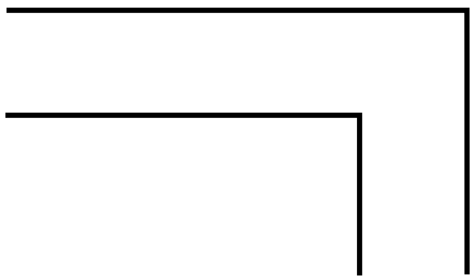
The Mathematical Removal Company 1

(by Alan Champneys)

The Mathematical Removal Company need to move a series of ladders from a window cleaning business through a corridor that is only 1 metre wide. The corridor consists of two long straight sections joined by a right-angled corner.



To being with, they need to calculate the longest ladder that can be manipulated around the corner, while keeping the ladder precisely horizontal. (You may ignore the thickness of the ladder.) Try drawing some pictures.

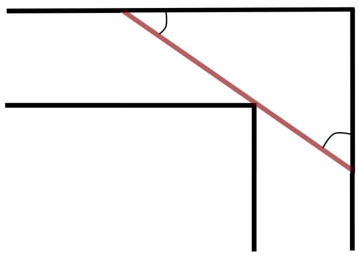
Imagine a ladder that is too long. Draw a picture of the point at which it gets stuck. What are the angles involved? Which direction can the ladder be moved?

Can you now work out what the angles would need to be if the ladder could be moved in either direction? So, what is the length of the longest ladder?

$$\ell = \qquad\qquad \text{metres}$$

Solution

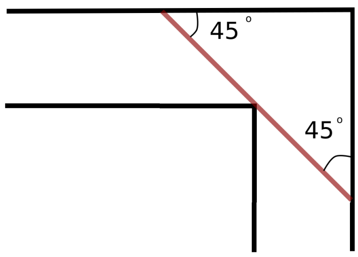
At the point a ladder gets stuck it touches at three points. Each end must be touching a wall and some point between these two ends will be touching the apex of the bend.



Drawing the angles we realise that, in general, the angle the ladder makes to one wall will be less than 45°, the other greater than 45°. Moreover, form Pythagoras' Theorem, the angles must add up to 90°.

But, in the drawing, we have not specified which directions the removers are trying to move the ladder.

Clearly, there is no problem moving the ladder away from the corner in the direction of the end making with the wall angle less than 45°. For example, dragging the ladder along that wall, the wall angle will decrease and the other end will detach from the opposite wall.



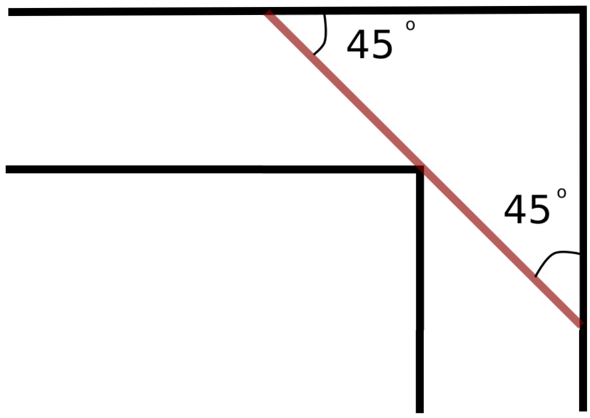
Taking this argument to its logical conclusion, we see that in the symmetric case where both angles are 45°, then the ladder can be moved in either direction. This then gives us the limiting configuration.

Simple application of Pythagoras' Theorem now gives the length of the ladder to be $2\sqrt{2}$ metres.

The Mathematical Removal Company 2

(by Alan Champneys)

You should have found something like the following diagram of how the longest possible ladder can fit around the bend.



This gives the maximum length of ladder that can be removed horizontally to

$$\ell_1 = 2\sqrt{2} = \text{ 2.82 metres}$$

More realistically, a professional removal worker would tilt the ladder into the third dimension in order to get it around a corner more easily.

So suppose the corridor has a uniform height of 3 metres. What is the maximum length of ladder that can be manipulated around the corner?

$$\ell_2 = \qquad\qquad \text{metres}$$

Solution

This problem is straightforward once you realise that the corner only occurs in the horizontal plane. In the vertical direction, the corridor is uniformly 3m high.

The best we can do is to tilt whatever fits in the plane so that it touches both the floor and the ceiling.

Then we can use Pythagoras' Theorem to obtain the longest ladder. So,

$$\ell_2^2 = \ell_1^2 + 3^2 = 8 + 9 = 17$$

So

$$\ell_2 = \sqrt{17} = 4.13\,\text{metres}$$

The Mathematical Removal Company 3

(by Alan Champneys)

The Mathematical Removal Company now want to decide on the optimal shape of box that can contain the most stuff to be removed, and still fits along the same corridor with the 90° bend. Assuming that the box is 3 metres tall so that it fills the entire height of the corridor, what should the cross-section of the box look like if:

the cross-section must be a square?;

the cross-section can be any rectangle?
[Hint: you can assume that the box can be rotate around the corner, and use the same idea as for the ladder.]

What do you notice about the cross-sectionaarea of A in each case?

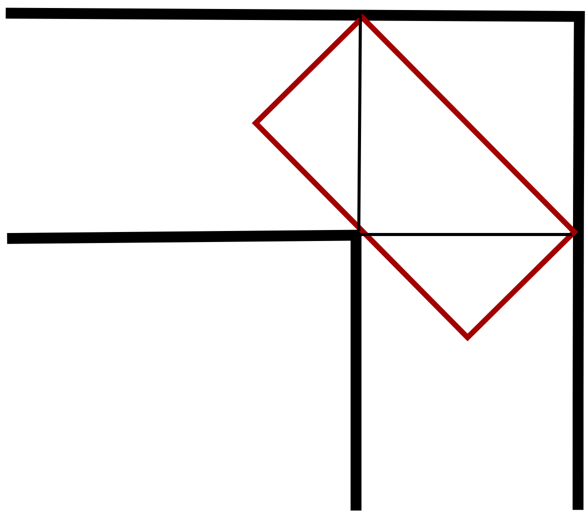
Solution

Clearly for the square box, the only thing that can be done is to slide the box up to the junction and then slide it away in the perpendicular direction, without rotation. Thus the largest square is

$$A_{\text{square}} = 1 \text{ metre } \times 1 \text{ metre } = 1\text{metre}^2.$$

Now, for a rectangle, consider the following diagram.

Appealing to the same symmetry argument as for the ladder, this is the largest rectangle that can be rotated around the corner.



Not that from Pythagoras' Theorem that the side lengths are $\sqrt{2}$ and $\sqrt{2}/2$. Therefore

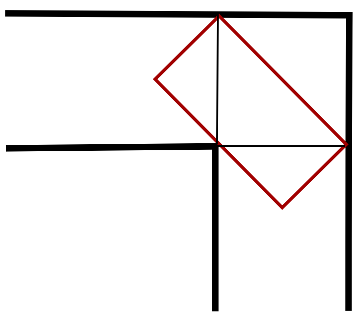
$$A_{rotatedrectangle} = \sqrt{2} \text{ metres } \times \frac{\sqrt{2}}{2} \text{ metres } = 1\text{metres}^2.$$

And so, the two areas are the same!

The Mathematical Removal Company 4

(by Alan Champneys)

You should have found, like the removers did, that the square box which is just shunted around the corner and the rectangular box that is rotated, have the same cross-sectional area of 1 metre^2. *[Note how the triangular areas of the rectangle that lie outside of the square fit perfectly into area above the rectangle to make the square.]*



One bright spark, the newest member of the Mathematical Removal Company's team, suggested they could carry more stuff if the boxes were semi-circular in cross section. Is she right?

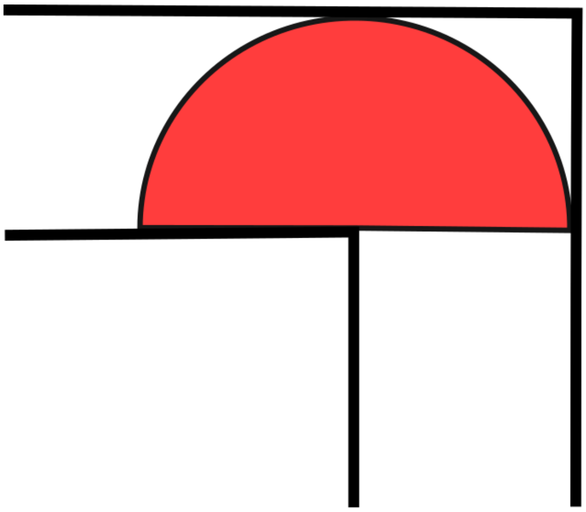
What is the largest semi-circular area that can be rotated around the corner?

Is this the largest possible area, or can you do even better? What is the largest cross-sectional area you can come up with?

[In fact this challenge is known as the Moving Sofa Problem,
see `https://en.wikipedia.org/wiki/Moving_sofa_problem`
and the solution to for the largest cross-sectional area not known]

Solution

Note how a semi-circle with its centre at the apex of the corner can be rotated around the bend in the corridor, provided the radius is less than the width of the corridor.



So the maximum radius is 1 metre. And the area is

$$\frac{1}{2}(\pi \times 1^2) = \frac{\pi}{2} \text{ metres}^2 = 1.57079 \text{ metres}^2.$$

The wikipedia page `https://en.wikipedia.org/wiki/Moving_sofa_problem` Shows a so-called Hammersley sofa, which is shaped like an old fashioned telephone handset.

To get students to think in the direction of finding this shape, suggest they try cutting a small semicircle out of the larger semicircle so that the shape can go around the corner more easily. Then you could ask what more can be added to the two ends of the shape.