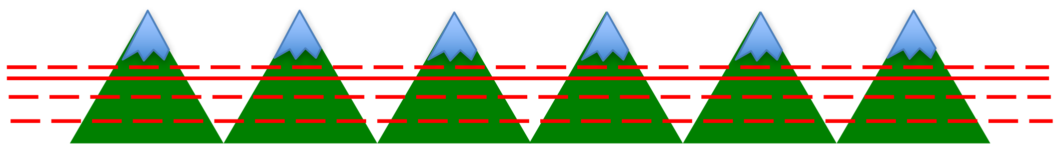
Railway line 1

(by Filippo Simini)

Design the path of a high-speed railway line across a mountain region in order to minimise the total construction cost. The construction costs depend on the excavation of tunnels and the construction of viaducts. The high speed trains should avoid slopes, so the railway line must run horizontally. Trains should also avoid sharp corners and travel as much as possible on a straight line. For simplicity, we can consider the problem in one dimension, where the mountains are a series of triangles placed next to each other and we can assume that the railway line is a straight horizontal line. The red lines in the figure below show possible railway lines under these assumptions.

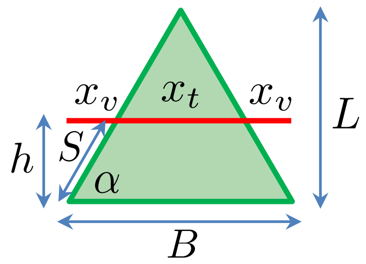


The cost of excavation of a tunnel can be assumed to be proportional to the length of the tunnel. For the cost of building a viaduct, we can consider two scenarios. In the first scenario, the cost is proportional to the viaduct length, as in the case of tunnels.

Can you find the optimal height of the railway line in this first scenario?

Solution

To find the height of the railway line, h, that minimises the total building cost we have to write the total cost, C, as a function of the height, $C(h)$. The total cost of the railway line is the sum of the costs to pass through each mountain. Since all mountains are equal, the total cost is proportional to the cost of one mountain, so it is enough to minimise the cost to build the railway line through one mountain.



The total cost is the sum of the cost of excavating tunnels and building viaducts:

$$C(h) = C_t(h) + C_v(h).$$

The cost of excavation of a tunnel is proportional to the tunnel's length, x_t: $C_t(h) = T \cdot x_t(h)$, where T is the excavation cost per unit length. In the first scenario, the cost of building a viaduct is proportional to the viaduct's length, $2x_v$, where x_v is the viaduct's length on each side of the mountain: $C_v(h) = V \cdot 2x_v(h)$, where V is the cost of the viaduct per unit length. So,

$$C(h) = T \cdot x_t(h) + V \cdot 2x_v(h)$$

Let's now find $x_t(h)$ and $x_v(h)$ as a function of h. For x_v we have $x_v(h) = S\,\cos\alpha$. Combining it with $h = S\,\sin\alpha$ we obtain

$$x_v(h) = h/\tan\alpha.$$

To find x_t we note that $B = x_t + 2x_v$. Since $B = 2L/\tan\alpha$ we obtain

$$x_t(h) = 2(L - h)/\tan\alpha$$

The total cost is

$$C(h) = \frac{2}{\tan\alpha}(T(L-h) + Vh).$$

From the formula it is clear that the height that maximises $C(h)$ is $h = 0$ if $T < V$, while it is $h \geq L$ if $T > V$. In other words, if it is cheaper to build tunnels the railway should be one long tunnel underneath all mountains, otherwise it should be one long viaduct above all mountains.

Railway line 2

(by Filippo Simini)

In the second scenario the cost to build a viaduct depends both on the length of the viaduct and on the height of its pillars, because more material is needed to build higher viaducts. So in this case we can assume that the cost is proportional to the area between the viaduct and the mountain.

Can you find the optimal height of the railway line in this second scenario?

Solution

In the second scenario, the cost of a viaduct is proportional to the area between the viaduct and the mountain, $2a_v$, where a_v is the area on each side of the mountain:

$$C_v(h) = V \cdot 2a_v = V \cdot 2\frac{h \cdot x_v(h)}{2},$$

where V is now the cost of the viaduct per unit area. Combining this with the cost of tunnels, which is the same derived in Question 1, we obtain the total cost:

$$C(h) = \frac{2}{\tan\alpha}(T(L-h) + Vh^2/2).$$

To find the maximum of this parabola we can differentiate $C(h)$ and find the h^* such that $C'(h^*) = 0$:

$$\frac{d}{dh}C(h)\mid_{h^*} = -T + Vh^* = 0 \qquad \Longleftrightarrow \qquad h^* = T/V.$$

Actually, given that the height must be smaller than L in order to minimise the cost, we have $h^* = \min(L, T/V)$. The minimum cost is then

$$C(h^* = L) = \frac{2}{\tan\alpha}(\frac{L^2V}{2})$$

if $T/V \geq L$, and

$$C(h^* = T/V) = \frac{2T}{\tan\alpha}(L - \frac{T}{2V}).$$

otherwise.

Note that if $T/V < L$ the optimum h^* is independent of α and L. This means that we can relax the assumptions of having mountains with the same slope and the same height (providing that the shortest mountain's height is larger than T/V) and the optimal railway's height will always be given by $h^* = T/V$.