Dimensionality, bias, independence and measurement scale problems in latent trait test score models

Harvey Goldstein

It is argued that latent trait models are in the analysis of test scores, in particular the Rasch model, suffer from serious deficiencies. They assume uniformly a measurement scale model for which little substantive justificiation seems to be available and they make the assumption of local independence, which means little empirical support. It is suggested that the individual ability estimates obtained are always confounded with the item effects. The Rasch model applied to binary data gives similar results to the item response models of Masters & Mokken (1974). However, these models do not allow item parameters to be used to obtain an ordering of the items. It is shown that the use of such models needs to be critically examined, especially in connection with dichotomous item response models using the item response model approach. The paper also discusses the dimensionality of the basic space, and shows how multidimensional models may be specified.

1. Introduction

This paper investigates some mathematical and statistical properties of certain latent trait score models, which seem to have been largely ignored or to have suffered from excessive specification. The applicability of these models in substantive areas such as education has been discussed elsewhere (see Goldstein & Blinkhorn, 1977; Goldstein, 1979). They have clear implications for such applications as education. It is shown that the Rasch model has clear assumptions of local independence and its relation to fitting models with one or more dimensional models is studied. The second section examines the usual assumption of local independence and its relation to fitting models with one or more dimensional models. The third section presents a method for fitting models with two or more dimensional models. The final section demonstrates that the usual estimates for the parameters are inherently biased. Throughout the paper, the simple logistic or logistic model known as the Rasch model (1960) will be the basis for discussion. This is because it has become one of the most popular and effective tools for test analysis. The paper will be divided into two parts. The first part will discuss the properties of the Rasch model, and the second part will present a method for fitting models with two or more dimensional models. The final section demonstrates that the usual estimates for the parameters are inherently biased.

2. Measurement scale models

The Rasch model can be written:

\[ P_{ij} = \frac{e^{\beta_j - \delta_i}}{1 + e^{\beta_j - \delta_i}} \]

where \( P_{ij} \) is the probability of the item response, \( \beta_j \) is the difficulty parameter for item \( j \), and \( \delta_i \) is the ability parameter for person \( i \). The Rasch model is a special case of the logistic regression model, where the logit of the probability is a linear function of the parameters. This means that the Rasch model is a very simple model, and it is often used as a baseline for more complex models, such as those that specify more than one parameter for each item in order to allow for different discriminating powers.
Problems in latent trait models

Problems in latent trait models arise primarily concerning the fixed effects version of (1), which has received some attention in the mixed effects model literature. An alternative model, which is more naturally associated with the fixed effects model, would be the random effects model, which is the model used in this paper. The random effects model has the advantage of not requiring the estimation of the fixed effects model, but it does make, typically, more assumptions than the fixed effects model. These assumptions are required to make the random effects model identifiable, and they are also required in many applications.

The random effects model is a more general model, and it can be used to model the relationship between the observed response and the latent variable. The latent variable is assumed to be normally distributed, and the observed response is assumed to be a linear function of the latent variable plus some random error. The random error is assumed to be normally distributed with mean zero and variance $\sigma^2$.

The relationship between the observed response and the latent variable can be written as:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

where $y_i$ is the observed response, $x_i$ is a vector of explanatory variables, $\beta_0$ and $\beta_1$ are the regression coefficients, and $\epsilon_i$ is the random error term.

The regression coefficients can be estimated using ordinary least squares (OLS) or maximum likelihood (ML) methods. OLS is a simple method that is easy to implement, but it assumes that the errors are normally distributed. ML is a more powerful method that can be used to estimate the variance of the errors, but it requires more computational effort.

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Figure 4. Illustration of the relationship between latent ability $\lambda$ and the probability of a correct response $P$. $P$ is a logit function of $\lambda$.

$P = \frac{1}{1 + e^{-\lambda}}$, where $x = \exp(\lambda)$.

The form of the relationship can be changed by the incorporation of additional factors, such as the influence of confounding variables. For example, if $P$ is a function of $\lambda$ and $x$, then the probability of a correct response can be written as:

$$P = \frac{1}{1 + e^{-\lambda - x}}.$$
\[
\begin{align*}
\frac{d}{dx} \left( \sqrt{1 + x^2} \right) &= \frac{x}{\sqrt{1 + x^2}} \\
\int \frac{x}{\sqrt{1 + x^2}} dx &= \ln \left| \sqrt{1 + x^2} + x \right| + C
\end{align*}
\]

In order to analyze the effects of applying a model with somewhat different properties, let us consider the effect of modifying the parameters of the model.

The key point here is that the modification of the parameters affects the overall behavior of the system. It is therefore important to carefully consider how these modifications will impact the results. The introduction of new parameters can lead to significant changes in the model's predictions, and it is crucial to understand how these changes will manifest in the system's behavior.

To illustrate the effect of applying a model with somewhat different properties, let us consider the following scenario:

- In the original model, the parameter \( p \) was fixed to a specific value, say \( p_0 \).
- In the modified model, the parameter \( p \) is allowed to vary within a certain range, say \( p_\text{min} \leq p \leq p_\text{max} \).

This modification changes the behavior of the system in a significant way. For example, the system may become more sensitive to changes in the input, or it may exhibit different patterns of behavior under varying conditions.

The implications of this modification are far-reaching. It highlights the importance of carefully considering how parameters affect the overall performance of a model. In many cases, even small changes in the parameters can lead to significant differences in the model's predictions and outcomes.

In conclusion, the introduction of new parameters can lead to substantial changes in the behavior of a model. It is therefore crucial to carefully consider how these changes will impact the system's performance and to validate the new model against existing data to ensure its reliability.
information mapping patterns avoid this to the same score. For example, the important
model may give a large value of attention to some parameters when referring to the
number of relevant images. The score of this model is high, even if there are
very few other relevant images. This problem is less common with models that
are trained with multiple models, each trained on a different set of images.

Another issue is the use of the attention scores with the score.

To improve some of these problems, a combination of two or more models will be
made to evaluate the importance of different features. For example, a model that
combines both image features and scene features will evaluate the importance of
different features in each model. This will allow the model to focus on the
attributes of the scene that are relevant to the task, rather than on the
attributes of the image that are not. This will also allow the model to
evaluate the importance of different features in different contexts, rather than
on a single context.

In addition, the model may have difficulties with some
features. For example, the model may have trouble
evaluating the importance of features that are not
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## 3. Local Independence

**Introduction:** This in general the same point that appears

In some of which are more potential names specific algorithms (see, for example, 

The main concern is expressed to approximate objects to a distribution. There are other 

In that situation, the point is that the oracle is used to estimate the model's 

Some models are simple and others are complex. Despite the differences in parameter 

Remark: The overall goodness of the selected model or composite.

By interactively determine some parameters, the model is used to maintain that 

For example, where there are several different classes, one may well want to examine a 

Note: The informal question is that the estimation of the relation for each item can be examined in the 

In Table I, the parameters determine the order and log-regression for the set of 

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
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<tbody>
<tr>
<td>Log-reg</td>
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**Table I.** Data point estimates for the order and log-regression for the set of
where $y_{i} \sim \mathcal{N}(\beta \cdot x_{i}, \sigma^2)$ is the joint distribution of responses $y_{i}$ for a given value of predictors $x_{i}$, and $\beta \cdot x_{i}$ is the marginal distribution for the ith individual. In the general multidimensional case, $\beta$ represents a vector of parameters.

If the $y_{i}$'s were independent, then the $y_{i}$'s are independent, and the covariance matrix of the responses would be $\sigma^2 I$, where $I$ is the identity matrix. However, in the case of non-independence, the covariance matrix is given by $\mathbf{C} = \sigma^2 \mathbf{R}$, where $\mathbf{R}$ is the correlation matrix of the responses.

If it were possible to estimate the parameters of the model, we could consider the joint distribution of the responses. However, if we have only the observed data, we can only estimate the marginal distribution of each response. The joint distribution of the responses is then given by the product of the marginal distributions, $f(y_1, y_2, ..., y_n) = \prod_{i=1}^{n} f(y_i)$. This is known as the product rule for probability distributions.

If we assume that the responses are independent, then the joint distribution is simply the product of the marginal distributions, $f(y_1, y_2, ..., y_n) = \prod_{i=1}^{n} f(y_i)$. However, if we assume that the responses are correlated, then the joint distribution will be different.

In the case of correlated responses, we can use the joint distribution to estimate the parameters of the model. However, if we have only the observed data, we can only estimate the marginal distribution of each response. The joint distribution of the responses is then given by the product of the marginal distributions, $f(y_1, y_2, ..., y_n) = \prod_{i=1}^{n} f(y_i)$. This is known as the product rule for probability distributions.
where the symbol $\mathcal{Q}$ denotes the set of all functions $f: \mathbb{R} \to \mathbb{R}$ that are absolutely integrable on $[a, b]$. For any $f, g \in \mathcal{Q}$, we define the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

The norm $\|f\|$ of a function $f$ is defined as

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

The space $L^2([-r, r])$ is a Hilbert space, and the key result is that the linear combinations of the Legendre polynomials are dense in $L^2([-r, r])$. This means that any function in $L^2([-r, r])$ can be approximated arbitrarily well by a linear combination of Legendre polynomials.

4. Dimensionality of the Legendre Space

In finite dimensions, the Legendre polynomials are orthogonal. However, when considering the entire space $L^2([-r, r])$, the orthogonality is only valid in a certain interval, typically $[-1, 1]$. This interval is crucial because it allows for the possibility of polynomial approximation within this range. Beyond this interval, the orthogonality relations break down, and additional considerations must be taken into account when constructing approximations.

To summarize, the Legendre polynomials provide a powerful tool for approximating functions in $L^2([-r, r])$, especially when the approximation is needed within the interval $[-1, 1]$. The orthogonality and completeness properties of these polynomials make them suitable for various applications in physics, engineering, and numerical analysis.
\[
\int_{-\infty}^{\infty} 
\frac{(e^x + 1)}{x} \, dx = \int_{-\infty}^{\infty} \frac{e^x}{x} \, dx = 0
\]

\[
\int_{-\infty}^{\infty} \frac{1}{x} \, dx = \int_{-\infty}^{\infty} \frac{e^x}{x} \, dx
\]

The information in these formulas is essential for applications in calculus. The integral expression for the function \(\frac{e^x}{x}\) over the infinite interval \([-\infty, \infty]\) is crucial in various mathematical and physical contexts. The result of 0 for the integral of \(\frac{1}{x}\) highlights a fundamental property of the Cauchy principal value used in complex analysis. These integrals are key in understanding the behavior of functions at infinity and in the study of asymptotic expansions.
The geometry means for the outer model

<table>
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<th>Parameter</th>
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Table 2: From different optimums the minimal individual utility estimates for the set of

condition of marginal independence is used (illustration in a real application).

The same geometric means for the outer model

L stands for the estimates of the $\theta$ together with the table of $\theta$. For $\theta = 1.1, \ldots, \theta_{0.6}$,

minimun individual utility estimates for the computation

and report models with the same geometric means. Although the absence of a

three invarient means with a different rate, a unique model can be created for the log-log

time of $\theta$ + 1 for any parameter $\theta$. A worth noting that the solution of $\theta = \theta_{0.1} \ldots, \theta_{0.6}$

which can be obtained for any model of $\theta$ from a knowledge of the value of the

$$
\left[ \int \frac{\theta_{0.1} - \theta_{0.2}}{\theta_{0.1} \theta_{0.2}} \right] \left[ \int \frac{\theta_{0.1} - \theta_{0.2}}{\theta_{0.1} \theta_{0.2}} \right] = |\theta_{0.1} - \theta_{0.2}|
$$

which is an arithmetic summation function of the $\theta_{0.1} \ldots, \theta_{0.6}$, required expected value of

$$
\prod_{\theta_{0.1} - \theta_{0.2}} \theta_{0.1} = 1
$$

where

$$
\left[ \int \frac{\theta_{0.1} - \theta_{0.2}}{\theta_{0.1} \theta_{0.2}} \right] \left[ \int \frac{\theta_{0.1} - \theta_{0.2}}{\theta_{0.1} \theta_{0.2}} \right] = |\theta_{0.1} - \theta_{0.2}|
$$

For the two model, we write $\phi_{0.1} \cdots \phi_{0.6}$ for the probability of obtaining

general not be equal to $\phi_{0.1} \cdots \phi_{0.6}$.
The ability factors, Blaws for sign boards at $1.5$ and $2$ geometry items for a range of
dispensations of total amount of fuel price for the set of two geometry items.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$E$</th>
<th>$10^6 d$</th>
<th>$10^8 d$</th>
<th>$E'$</th>
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<td>6.0</td>
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<tr>
<td>11.1</td>
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<td>15.0</td>
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\[ E = \left( \sum_{i=1}^{n} e_i \right)^{\frac{1}{n}} \]
Discussion

Sample

The hierarchy of factors that influence the performance of different models is an important aspect of our analysis. The models that we have developed differ in the way they incorporate the different factors into their predictions. In the simplest case, the models are based on a set of linear equations that relate the factors to the predictions. In the more complex cases, the models incorporate non-linear interactions between the factors.

Instead of relying on a single model, we have developed a hierarchy of models that allows us to explore the different factors in a systematic way. The models are based on a set of equations that relate the factors to the predictions, and the hierarchy of models allows us to explore the different factors in a systematic way.

The factors that influence the performance of the models are shown in the table below. The factors are arranged in order of importance, with the most important factors at the top of the hierarchy. The hierarchy of factors is important because it allows us to explore the different factors in a systematic way.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Importance</th>
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</thead>
<tbody>
<tr>
<td>Factor A</td>
<td>High</td>
</tr>
<tr>
<td>Factor B</td>
<td>Medium</td>
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<tr>
<td>Factor C</td>
<td>Low</td>
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