Identifying Behavioral Types

Christopher Kops, Paola Manzini,
Marco Mariotti, Illia Pasichnichenko
Discussion Paper 26/822

11th February 2026

School of Economics

University of Bristol
Priory Road Complex
Bristol
BS8 1TU
United Kingdom

University of BRISTOL
School of Economics

Identifying Behavioral Types

Christopher Kops, Paola Manzini, Marco Mariotti, Illia Pasichnichenko*

February 11, 2026

Abstract

We study identification in models of aggregate choice generated by unobserved behavioral types. An analyst observes only aggregate choice behavior, while the population distribution of types and their type-level choice patterns are latent. Assuming only minimal and purely qualitative prior knowledge of the process generating type-level choice probabilities, we characterize necessary and sufficient conditions for identifiability. Identification obtains if and only if the data exhibit sufficient cross-type behavioral heterogeneity, which we characterize equivalently through combinatorial matching conditions between types and alternatives, and through algebraic properties of the matrices mapping type-level to aggregate choice behavior.

JEL codes: D81, D83, D91

Keywords: revealed preferences, stochastic choice, bounded rationality, attention, characteristics, incomplete preferences.

1 Introduction

Latent factors such as preferences, attention, and cognitive capacity influence choice behavior. These factors are typically heterogeneously distributed—be it across individuals in a population or within a single decision maker observed repeatedly over time. This heterogeneity then gives rise to various *behavioral types*. Whenever the individual choices of these types remain unobservable—for example because of privacy protection, anonymization, or the sheer scale of data—they may still produce observable choice data on the aggregate level. This raises a fundamental question: when is it possible to infer the distribution of behavioral types, or their type-specific choice patterns, from aggregate choice data alone? This paper

*Kops: Maastricht University, the Netherlands (e-mail: j.kops@maastrichtuniversity.nl); Manzini: University of Bristol, United Kingdom (email: p.manzini@bristol.ac.uk) and IZA Bonn; Mariotti: Queen Mary University of London, United Kingdom (email: m.mariotti@qmul.ac.uk) and Deakin University; Pasichnichenko: University of Sussex, United Kingdom (email: i.pasichnichenko@sussex.ac.uk). We thank audiences at Sussex, Maastricht, Karlsruhe, the 11th BRIC Conference at ITAM, and the 2025 Unforeseen Contingencies Workshop at UGA.

characterizes conditions under which such inferences can be drawn, assuming minimal and purely qualitative prior knowledge of the process generating type-level choice probabilities.

To illustrate the motivation, consider a policymaker contemplating a stimulus that expands the set of options available to consumers. Assessing the effectiveness of such a policy requires knowledge of the distribution of consumer types. Some types may attend to the full range of available options, others may focus only on a subset, and still others may consistently adhere to their current choice. The overall impact of the policy depends both on the prevalence of these behavioral types and on the manner in which each type responds to the expanded choice set.

We study a general framework of stochastic choice. The analyst observes choice probabilities (interpreted as idealized empirical *shares*) over a finite set of n alternatives. These shares result from the aggregation of the unobserved stochastic choices made by a finite number of behavioral types. Formally, the core framework is captured by the equation

$$p = M\pi \tag{1}$$

where p is an observed $n \times 1$ vector of choice probabilities over the alternatives listed in a given order, and M is an $n \times r$ matrix describing the partially unknown choice probabilities of r *types*. A type t is a probability distribution over the alternatives. The entry $M(k,t)$ of the matrix M denotes the probability with which type t chooses alternative k. A theory of choice specifies how types choose, that is, the entries in M. Finally, π is an unobserved $r \times 1$ vector of probabilities expressing the distribution of types. Since often stochastic choice data are observed in multiple occasions of choice, we also consider an extension of the model in Equation (1) in this direction.

A key feature of our approach is that we impose only minimal informational requirements on the analyst's knowledge of the choice process. Specifically, the analyst is assumed to have prior knowledge only of the *possibility relation* associated with type-level choice behavior—namely, which types may choose a given alternative with positive probability and which types cannot possibly do so. Such information is often implicit in the definition of behavioral types in given contexts: vegan consumers do not choose animal products; risk-averters steer clear of junk bonds; information avoiders omit health screenings or ignore adverse media sources. At a theoretical level, behavioral models may similarly restrict the support of type-level randomization, for instance by allowing randomization only within a consideration set or among maximal elements of an incomplete preference relation.[1]

Our focus is on how the data in p can be used to identify the unknown parameters—

[1]Concrete examples are studied in more detail in Section 6.

namely, the elements of π or the positive elements of M.[2] Under the informational assumption discussed above, we derive identification results in three settings. First, we characterize identification of the type distribution π without imposing further restrictions on the structure of type-level choice behavior.

Second, we specialize the framework to a type–state model, in which the stochastic component of each type's behavior depends on the realization of a common random state that is unobserved by the analyst. In this setting, aggregate choice probabilities arise as weighted sums of state-dependent choice matrices, according to the equation

$$p = \left(\sum_{a \in A} f(a) M^a\right) \pi$$

where A is a set of states, $f(a)$ is the probability of state a, and M^a is a matrix of $0-1$ type-level choice probabilities that applies in state a. In this model, we obtain a further set of conditions for the identification of π.

Finally, in Section 5, we demonstrate that observing choices across multiple occasions strengthens identification, as it allows the analyst to exploit cross-temporal variation. For this purpose, we combine the new one-shot identification conditions with a fundamental result from tensor decompositions. This leads to sufficient conditions for the identification of both the type distribution and the type-level choice probabilities.

Our identification results can be broadly understood as formalizing a requirement of sufficient *behavioral heterogeneity*. They fall into two main classes. The first class provides a combinatorial characterization, based on the existence of suitable *matchings* between types and alternatives that are consistent with the possibility relation. These matchings capture heterogeneity in the sense that different types can, in principle, be associated with distinct choices when there is sufficient behavioral richness. The second class provides a geometric and algebraic counterpart to this analysis. We establish new results on the generic invertibility of weighted sums of state-dependent choice matrices, formulated as restrictions on the nullspaces of the matrices M^a. These algebraic conditions clarify *why* identification failures arise—by making explicit the directions in aggregate choice space along which cancellations occur—and admit a natural interpretation as requirements of behavioral heterogeneity.

The simplest case—consisting of a single choice occasion, two states a and b, two types, and two alternatives—already illustrates the multifaceted nature of our results. In this setting, generic (i.e., holding for all parameter values except possibly on a measure-zero set) identifiability of the type distribution is equivalent to either of the following conditions: (i)

[2]In the next section we clarify the notion of genericity used in the paper.

the nullspaces of M^a and M^b intersect only trivially; (ii) in at least one of the two states, there exists a matching between types and alternatives. The behavioral meaning of both conditions is that at least one state must "separate" the two types, in the sense that the types choose differently in that state. If, in addition, each type's behavior is constant across states, then identification becomes global—i.e., it holds for all parameter values rather than merely generically.

Overall, our results provide an exhaustive characterization of the scope for identification. It is important to emphasize that the notion of behavioral heterogeneity across types at the root of this characterization is purely *qualitative*: our identifying conditions do not depend quantitatively on the type-level choice probabilities, but only on the pattern of structural zeros in choices. Equivalently, the entire analysis depends solely on the sign pattern of the matrix M.

The scope and applicability of our results is demonstrated by two extended examples in Section 6. The first examines choice between vectors of characteristics *à la* Lancaster [27], while the other studies a model of incomplete preferences.

1.1 Related literature

Dardanoni *et al.* [13] has a similar motivation to ours, in that it studies generic type identification from discrete choice data generated by a single menu without additional covariate information, linking identifiability to the invertibility of an associated matrix. However their analysis is concerned with a specific parametric model of choice with consideration sets, in which types are indexed by consideration capacities—that is, by the maximum number of alternatives a decision maker can consider. In that setting, the matrix representation of the model has a known and highly structured form. A more econometrics-facing work in the same vein, which however makes use of covariates besides choice observations from a single menu with limited consideration (or limited feasibility) is Barseghyan *et. al.* [6]. By contrast, the framework developed here is deliberately abstract and qualitative. Our objective is not to analyze a particular behavioral model, but to accommodate a broad class of models as special cases and to characterize identification at this level of generality and exploiting only the qualitative nature of the process of choice.

A recent strand of the literature studies the identifiability of the preference distribution from choice data within Random Utility Maximization (RUM) frameworks. Contributions in this vein include Apesteguia *et al.* [3], Chambers and Turansick [11], Doignon and Saito [15], Filiz-Ozbay and Masatlioglu [18], Lu [29], Manzini and Mariotti [31],[3] Turansick [37], and

[3]See also the Correction in Manzini *et al.* [32].

Yildiz [38] (see also the important earlier piece by Fishburn [19]). Our model shares with this literature a broad objective within a stochastic choice framework, namely the identification of underlying heterogeneity from observed choices. Our analysis departs from this literature in two main respects. First, we do not take preferences, rankings, or general choice functions as primitives; instead, we work with abstract "behavioral types." Second, standard RUM approaches typically take as observable a random choice rule specifying a distribution of choices from each of several overlapping menus—distinct subsets of a common grand set. By contrast, the core of our analysis relies on the observed choice distribution from a single set of alternatives, although we also provide an extension to a parsimonious multi-occasion framework. These differences necessitate distinct analytical tools and arguments.

Caradonna and Turansick [10], like the papers above, also focus on RUM identification, but this is mainly for expositional purposes. Their work develops new general techniques for identification (related to the unique decomposition of flows on directed acyclic graphs into measures over paths) that extend to more general random choice models, as well as dynamic discrete choice models. Once again, a main difference from our approach lies in their use of a random choice rule as a primitive.

Progress on identification with more limited data than a full random choice rule can be credited to Azrieli and Rehbeck [5]. Their framework is one of *marginal stochastic choice*, a setting in which only the probability of choosing each alternative and the probability of each menu's realization are observed.

Our multi-occasion extension of the main framework shares methodological affinities with Allman *et al.* [1] and Dardanoni *et al.* [14], which in turn build on classical tensor decomposition results by Kruskal [25], with a recent proof by Rhodes [35].

At a much broader level our work also relates to latent structure models, running from finite mixture models (McLachlan and Peel [34]) to nonparametric mixture models (Lindsay [28]), and hidden Markov models (Cappé *et al.* [9], Ephraim and Merhav [17]). The identification problem of mapping observed choice data back to latent behavioral types finds its formal roots in the seminal contributions of Koopmans and Reisersøl [22] and Koopmans [23]. Traditionally, the study of identifiability assumes a hypothetical, exact knowledge of the distribution of observables. The central question then is whether the underlying structural parameters can be uniquely recovered from this distribution. While such identification is distinct from statistical inference, it remains a prerequisite for it, as non-identifiable parameters render consistent estimation impossible.

While we focus entirely a on a qualitative notion of behavioral heterogeneity, a complementary approach is taken by Apesteguia and Ballester [2], who study the *measurement* of behavioral heterogeneity in environments closely related to those underlying our identification analysis.

2 Framework and definitions

Let $T = \{t_1, \ldots, t_r\}$ and $X = \{x_1, \ldots, x_n\}$ be finite sets of types and alternatives, respectively. The analyst observes choice shares p for all alternatives in X. Let M be an $n \times r$ matrix with a typical column containing the choice shares for a specific type, so that the data can be described by Equation (1).

As noted, similarly to the RUM framework, this model admits both a population and an intrapersonal interpretation. Under the population interpretation, type heterogeneity captures variation in preferences or cognitive traits across individuals. Under the intrapersonal interpretation, the types instead represent distinct choice-relevant states—preferential or cognitive—in which a single decision maker may find themselves at different points in time.

Definition 1. A type t_l is *possible at alternative* x_k if $M(k, l) > 0$. In this case we write $t_l \wedge x_k$, where $\wedge$ is a binary relation over the sets X and T.

The possibility relation $\wedge$ captures the analyst's prior knowledge about the structural zeros of the type–conditional choice matrix M, while leaving unrestricted the magnitudes of the positive entries. Thus, we identify the parameter space of the model with a full dimensional subset of $[0, 1]^{|\wedge|-1}$, where $r - 1$ dimensions describe π and $|\wedge| - r$ dimensions describe nonzero entries of M. The model's parameterization map is defined on the parameter space and takes values in $[0, 1]^n$. The model is *globally identifiable* if its parameterization map is injective. Often we will refer to identifiability as a property of (a subset of) the parameters in the model.

In this paper we focus mostly on *generic identifiability*. This implies that the set of parameter values for which identifiability does not hold has measure zero. In this sense, generic identifiability is often sufficient for data analysis purposes. We next briefly lay out the notions from algebraic geometry used repeatedly throughout the paper. For more thorough introductions to the field, see Allman *et al.* [1] and Cox *et al.* [12].

An algebraic variety is defined as the simultaneous zero-set of a finite collection of multivariate polynomials $\{\phi_i\}$ on $\mathbb{R}^m$. A variety is all of $\mathbb{R}^m$ only when all ϕ_i are 0; otherwise, a variety is called a proper subvariety and must be of dimension less than m, and, hence, of Lebesgue measure 0 in $\mathbb{R}^m$. The same is true if we replace $\mathbb{R}^m$ by a full-dimensional subset of $[0,1]^m$. Intersections of algebraic varieties are algebraic varieties as they are the simultaneous zero-set of the unions of the original sets of polynomials. Finite unions of varieties are also varieties, since if sets G_1 and G_2 define varieties, then $\{fg | f \in G_1, g \in G_2\}$ defines their union. Given a set $\Theta \subseteq \mathbb{R}^m$ of full dimension, we say that a property holds *generically* on Θ if it holds for all points in Θ, except possibly for those on some proper subvariety. Thus, generic identifiability of parameters means that all non-identifiable parameter choices lie within a proper subvariety, which is a set of Lebesgue measure zero.

Finally, the model with parametrization map z is *structurally non-identifiable* if no point in Θ has a singleton fiber, that is, for all $\theta \in \Theta$, $|\mathcal{F}(\theta)| > 1$, where $\mathcal{F}(\theta) = \{\theta' \in \Theta | z(\theta') = z(\theta)\}$ is the fiber of z at θ.

3 Characterizing identifiability of the type distribution

In this sparse observational context, our primary aim is the identification of π. We will show that generic identification is equivalent to the types' choices being sufficiently heterogeneous in a qualitative sense. Specifically, we establish a connection between the identifiability of π and the possibility relation $\wedge$.

The following notion describes the possibility for types to choose distinct alternatives.

Definition 2. A (type-to-alternative) *matching* is an injective function $m : T \to X$ such that $t \wedge m(t)$ for all $t \in T$.

In our setting, identification of π in the model defined by Equation (1) reduces to whether the structural zero pattern encoded by $\wedge$ admits parameter configurations for which the columns of M are linearly independent. The first result shows that this condition admits a simple combinatorial interpretation in terms of matchings between types and alternatives:

Theorem 1. *The parameters in π are generically identifiable if and only if there exists a matching $m : T \to X$. In addition, if a matching exists and it is unique, then the parameters in π are globally identifiable. If no matching exists, then the parameters in π are structurally non-identifiable.*

Proof. Please refer to Appendix A.1. □

Note that the existence of a matching implies $n \geq r$, i.e., there are at least as many alternatives as types. The interpretation of the matching is that r agents of distinct types can always choose r distinct alternatives. In this case, the possibility relation $\wedge$ is sufficiently rich to permit full identification of the type distribution from observed choices. Theorem 1 shows that the existence of a matching restricts the admissible behavioral theories exactly to those in which identification arises almost always. The theorem also clarifies that the failure of identification on measure zero sets is attributable to the multiplicity of matchings: when the matching is unique, identification occurs for all values of the parameters. As we will show below, the converse is not true.

The necessity of a matching can be understood as follows. If no matching exists between types and alternatives, then there is a subset of types whose "collectively choosable" alternatives—that is, all those at which one or more types in the subset are possible—are too few to accommodate them all. As a result, any attempt to assign distinct alternatives to these types must necessarily assign at least two types to the same alternative, or assign some type to an alternative it can never choose. Translating this into matrix terms, the columns of M corresponding to these types must be supported on a set of rows that is not rich enough to ensure generic linear independence. This issue is *structural*: it does not depend on the particular values of the positive entries of M, but follows solely from the zero pattern implied by the possibility relation. No selection of the positive parameter values in M can, in this case, permit learning of π from the data.

Remark 1. Formally, the notion of a matching used here is equivalent to that of a perfect matching in a bipartite graph whose left and right vertices correspond to columns and rows of the matrix M, respectively, with edges induced by its nonzero entries. Hence, the existence of a matching is equivalent to the condition in Hall's marriage theorem (Hall [21]). Translated to our context, the condition requires that for any set $S \subseteq T$ of types there exist at least $|S|$ distinct alternatives such that some type in S is possible at at least one of them. This further clarifies the sense in which the existence of a matching captures behavioral heterogeneity.

The following examples illustrate the result.

Example 1. Suppose that there are three types, labeled t_1, t_2, and t_3, and three alternatives, labeled x, y, and z. A behavioral theory determines the possibility relation $\wedge = \{(t_1, x), (t_2, x), (t_2, z), (t_3, x), (t_3, y), (t_3, z)\}$. In turn this induces the zero entries of M:

$$M = \begin{array}{cc} & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} {\color{red}\bullet} & \bullet & \bullet \\ 0 & 0 & {\color{red}\bullet} \\ 0 & {\color{red}\bullet} & \bullet \end{pmatrix} \end{array},$$

where black dots ($\bullet$) and red dots (${\color{red}\bullet}$) denote parameters. This encapsulates the information on which alternatives could possibly have been chosen by which types. The parametric description of this matrix takes values in $[0,1]^3$. It is clear that the columns of M can never be linearly dependent, whatever the values assigned to the dots. To build a matching, observe that t_1 must be matched to x, because this is the only alternative for which this type is possible. Then, t_2 must be matched to z and t_3 to y. The constructed matching (in red) is unique. On the other hand, if type t_1 were possible at y, so that the entry $M(2,1)$ is positive, there would exist other matchings, such as t_1 matched to y, t_2 to x and t_3 to z. In this case the columns could be linearly dependent for some values of the parameters, but the set of such values is of measure zero.

Example 2. Suppose that each type coincides with a consideration capacity, as in Dardanoni et al. [13], namely the maximum number of alternatives the agent can consider. Let $T = \{t_1, t_2, ..., t_n\}$ with type t_k having consideration capacity k.[4] Each type maximizes on the set of considered alternatives a known and common preference, namely a linear order $\succ$ defined on X. Type t_k considers each subset of X of cardinality k with a probability governed by an unknown full-support probability distribution on $\{K \subseteq X|\ |K| = k\}$. This model generalizes the fixed-preference model in [13], who assume the distribution on each $\{K \subseteq X|\ |K| = k\}$ to be uniform.

To build a matching, number the alternatives $x_1, x_2, ..., x_n$ in decreasing order of preference. Observe that type t_n, who considers the entire set X, chooses x_1 with probability one and must therefore be matched to x_1. Type t_{n-1}, who considers all but one alternative, chooses with positive probability only x_1 and x_2, so that, in view of the previous matching, must be matched with x_2. Continuing in this way, we build a (unique) matching, with type t_k matched to x_{n-k+1} for each k. It follows from Theorem 1 that the type distribution is identified. Proposition 2 in [13] is a special case of this result. [5]

3.1 Global identifiability

To provide a full characterization of global identifiability, we introduce some additional standard terminology. For any given permutation σ on the index set $\{1, 2, ..., r\}$, an inversion is a pair (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$. A permutation is *even (resp., odd)* if it consists of an even (resp. odd) number of inversions. Any matching $m : T \to X$ is associated with a permutation σ_m defined by $m(t_i) = x_{\sigma_m(i)}$. The *parity* of a matching m is defined as *even*

[4] Here, type t_n stands for a composite type including all types that can consider *at least* n alternatives.

[5] What is more, Proposition 6 in [13], showing identifiability for generic preference distributions in the extension of their model to heterogeneous preferences, is also implied by Theorem 1—note that the uniform distribution assumption on equal-sized consideration sets makes type-level choice probabilities structurally fixed for any given preference, so that the only variable parameters in this model are those that relate to the preference distribution.

(resp., *odd*) if σ_m is an even (resp., odd) permutation.[6] Two matchings have the same parity if and only if one can be transformed into the other by an even number of pairwise swaps of assignments between types. More precisely, a swap is a permutation that exchanges two elements and leaves all the others fixed, that is for distinct indices $i \neq j$ the swap between i and j is the permutation σ such that $\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(k) = k$ for all $k \neq i, j$.

The following result shows that the transition from generic to global identifiability can be obtained by restricting the way multiple matchings can coexist.

Theorem 2. *The parameters in π are globally identifiable if and only if there exists a subset $R \subseteq X$ with $|R| = r$ such that:*

(i) There exists a matching $m : T \to R$;

(ii) Any other matching $m' : T \to R$ can be obtained from m via an even number of swaps.

Proof. Please refer to Appendix A.2. □

The condition in Theorem 2 coincides with the classical parity characterization of sign-nonsingular (SNS) matrices. Our result simply adapts this characterization to a rectangular and column-stochastic setting, and interprets it as a condition for global identifiability of the type distribution.[7] In this interpretation, a matching is a possible "explanation" of the data imputing the choice of an alternative to an exclusive type. The difficulty with two matchings connected by an odd number of swaps is that they allow a complete reassignment of alternatives across types that leaves aggregate choice behavior unchanged. Such a reassignment creates two distinct but observationally equivalent explanations of the data, undermining global identifiability.

When there exists a unique matching, condition (ii) in the statement is vacuously satisfied; therefore, Theorem 2 implies as a special case the sufficient condition for identifiability of Theorem 1.

4 The type-state framework

We now endow the framework with additional structure, which is common in many economic environments. This allows for a more nuanced identification analysis than that de-

[6] While the parity of a matching depends on the chosen orderings of types and alternatives, only relative parity between matchings matters for our results, so the choice of ordering is immaterial.

[7] SNS matrices are square matrices where every real matrix with the same sign pattern (positive, negative, or zero) is nonsingular. They were introduced to Economics by Samuelson [36], and later studied by others (e.g. Bassett *et al.* [7], Gorman [20], Lancaster [26]), not in connection with identification but rather with comparative statics and the correspondence principle. For a modern mathematical treatment of SNS matrices see Brualdi and Shader [8], Chapter 6.

veloped in the previous section. We assume that the realization of a stochastic state determines the nature of the matrix M, while choice by each type is deterministic conditional on the state. Denote by f a probability measure defined on a known finite set A of states. Each type t first observes a realization of a according to f. Then, t chooses according to a "choice function" $c_t : A \to X$, where $c_t(a)$ denotes the choice made by type t in state a. In this sense, f is independent from the type.

For a given state $a \in A$ denote by M^a the corresponding $n \times r$ matrix of choice shares for each type. A column l of this matrix is a one-hot vector such that $M^a(k,l) = 1 \iff c_{t_l}(a) = x_k$. The matrix M is a convex combination of such matrices,

$$M = \sum_{a \in A} f(a) M^a. \tag{2}$$

In this way, a parametric description of M is given by $\{f(a)\}_{a \in A}$ taking values in $[0,1]^{|A|-1}$. Here is a simple example of this structure:[8]

Example 3. Suppose that each state a corresponds to a linear preference order $\succ_a$ over X, and that $f(a)$ gives the probability assigned to preference $\succ_a$. Types differ in their *attention depth*. Specifically, alternatives are ordered by salience and type t_k searches only the k most salient alternatives. The columns of M^a describe the type-level choices of agents with a given attention depth. Index alternatives in decreasing order of salience. Then the choice function of type c_{t_k} is determined according to $c_{t_k}(a) = \arg\max_{\succ_a} \{x_1, ..., x_k\}$, for $k = 1, ...n$. That is, for each state a, type t_k chooses the most-preferred alternative among the first k alternatives according to $\succ_a$. The resulting state-aggregated choice probabilities in M are then given by Equation (2).

We establish two classes of characterization results that guarantee generic identification. The first class provides matching conditions between types and alternatives, in the spirit of those developed in the previous section. The second class elucidates the *sources* of identification failure by focusing on algebraic properties of the matrix M—in particular, conditions involving the nullspaces of its constituent matrices—that determine its generic invertibility and their behavioral interpretation.

4.1 Identifiability of the type distribution via matching

The following definition formalizes the idea of a state in which the type is revealed through choice.

[8]More elaborate examples are studied in Section 6.

Definition 3. A state $a^* \in A$ *separates* types if, for any two types t_1 and t_2, we have $c_{t_1}(a^*) = c_{t_2}(^*)$ only if $t_1 = t_2$.

Thus, "a^* separates types" means that in state a^* there is a matching of the types with the chosen alternatives. When a state does not separate types we say that it *pools* them. The following result shows that, within the type-state framework, the existence of just one type separating state is sufficient for generic identification. Although this result follows from our later characterization in Theorem 4, it is of independent interest due to the simplicity of its condition.

Theorem 3. *Suppose there exists a state $a^* \in A$ that separates types. Then the parameters in π are generically identifiable.*

Proof. Please refer to Appendix A.3. □

To illustrate the previous result, consider the following example.

Example 4. There are three types, three alternatives, and two states, $A = \{a, b\}$, and

$$M^a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M^b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & f(b) \\ 0 & 0 & f(a) \\ 0 & 1 & 0 \end{bmatrix},$$

where the third matrix is defined by $M = f(a)M^a + f(b)M^b$. Clearly, a separates types but b does not. Because $f(b) = 1 - f(a)$, the complete parameter description of M is given by $f(a) \in [0, 1]$. By Theorem 3, the parameters in π are generically identifiable. Indeed, we can see that π is not identifiable only for $f(a) = 0$.

We can make further progress by specifying the definition of a matching for the current case.

Definition 4. A *matching* is an injective function $m : T \to X$ such that for all $t \in T$, $m(t) = c_t(a)$ for some $a \in A$.

Note that when A consists of a single state a, the existence of a matching is equivalent to the fact that a separates types. The type-state framework also allows us to record the states in which each type is matched to its corresponding alternative, motivating the following definition.

Definition 5. A *state-matching* is a pair (m, γ) where m is a matching and $\gamma : T \to A$ is such that $m(t) = c_t(\gamma(t))$ for all $t \in T$.

In words, a state-matching is defined by (i) an exclusive assignment of an alternative to each type; and (ii) a specification of the state in which any given type selects the assigned alternative. Note that for a single matching m there may be various state-matchings (m, γ), (m, γ'), etc.

For any state-matching (m, γ), let $\Gamma(m, \gamma) = \left[\nu_1 \ldots \nu_{|A|}\right]$ be the state-usage vector, indicating the numbers of times each state a_i is used to form the matching m, i.e.,

$$\nu_i = \sum_{t \in T} \mathbf{1}\left[\gamma(t) = a_i\right].$$

The following result spells out a condition which is both necessary and sufficient for the generic identification of π in the type-state framework. For any subset of alternatives $R \subseteq X$ with $|R| = r$, let M_R denote the $r \times r$ submatrix of M obtained by retaining the rows associated with R (and all r columns). Denoting $\operatorname{Im}(m) = \{m(t) \mid t \in T\}$ the image of m, let

$$\mathcal{S}_R := \{(m, \gamma) : (m, \gamma) \text{ is a state-matching and } \operatorname{Im}(m) = R\}.$$

Theorem 4. *The parameters in π are generically identifiable if and only if there exists a subset $R \subseteq X$ with $|R| = r$ such that the following two conditions hold:*

(i) $\mathcal{S}_R$ is nonempty.

(ii) For any partition of $\mathcal{S}_R$ into disjoint pairs $\big((m, \gamma), (m', \gamma')\big)$ with $\Gamma(m, \gamma) = \Gamma(m', \gamma')$, there exists at least one pair for which m' can be obtained from m via an even number of swaps.

Moreover, if no such R exists, then π is structurally non-identifiable.

Proof. Please refer to Appendix A.4. □

The first condition in Theorem 4 simply asserts the existence of a state-matching (m, γ) for which the image of m is the subset of alternatives R. The second condition is a restriction on how state-matchings in $\mathcal{S}_R$ of different parities can coexist within a given state-usage profile. In particular, it forbids the possibility that state-matchings of opposite parity be paired in such a way that every pair is identical in terms of its state usage. Note that this condition weakens the one in Theorem 2, which required *all* matchings to be obtainable from each other via an even number of swaps. In the proof, the first condition ensures that at least one nonzero multiplicative term appears in the Leibniz expansion of the determinant, while the second guarantees that these terms are not fully cancelled at the summation stage. Theorem 9 below will offer a behavioral interpretation and an equivalent algebraic condition.

The next two examples show situations where, respectively, the conditions of Theorem 4 are or are not satisfied.

Example 5. Consider

$$\alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha & \alpha \\ \beta & 0 & \beta \end{bmatrix},$$

where $X = \{x, y, z\}$, $A = \{a, b\}$, $\alpha = f(a)$, and $\beta = f(b)$. The columns of the right-hand side matrix are generically independent because the determinant is not zero, for example, for $\alpha = \beta = \frac{1}{2}$. There is a state-matching $(m, \gamma) = (xyz, aab)$, with $\Gamma(m, \gamma) = [2\ 1]$. There is another state-matching $(m', \gamma') = (zxy, bba)$, with $\Gamma(m', \gamma') = [1\ 2]$. Clearly, there is no other state-matchings. Since $\Gamma(m, \gamma) \neq \Gamma(m', \gamma')$, the second condition is vacuously satisfied. Note that Theorem 3 would fail in this case because neither state separates types.

Example 6. The following is an instance where the second condition of Theorem 4 does not hold:

$$\alpha \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & \alpha & 0 \\ 0 & \alpha & 0 & \alpha \\ \beta & \beta & 0 & 0 \\ 0 & 0 & \beta & \beta \end{bmatrix},$$

where $X = \{x, y, z, w\}$. The columns of the right-hand side matrix are linearly dependent for all α and β, because the sum of the first and fourth columns equal to the sum of the second and third column. To see how this case is ruled out in Theorem 4, note that there is one state-matching $(m, \gamma) = (xzwy, abba)$, with $\Gamma(m, \gamma) = [2\ 2]$, and another state-matching $(m', \gamma') = (zyxw, baab)$ with $\Gamma(m', \gamma') = [2\ 2]$. Observe that m' is obtained from m by three swaps. It is easy to check tha there does not exist any other state-matching. Hence, the second condition of Theorem 4 is not satisfied.

Clearly, Theorem 3 follows from Theorem 4, since type separation by some $a^* \in A$ implies that there is a state-matching such that $\gamma(t) = a^*$ for all $t \in T$, with no other state-matchings having the same Γ.

It is also instructive to examine the following simplest $2 \times 2 \times 2$ case:

Example 7. Consider the type–state framework with two types $T = \{t_1, t_2\}$, two states $A = \{a, b\}$, and two alternatives $X = \{x_1, x_2\}$. Suppose, towards a violation of the identifiability condition, that both states pool types—for instance, $c_{t_1}(a) = x = c_{t_2}(a)$ and $c_{t_1}(b) = y = c_{t_2}(b)$. Then there exist two symmetric state-matchings: $(m, \gamma) = (xy, ab)$ and $(m', \gamma') = (yx, ba)$. In both state-matchings, each state is used exactly once and m' is obtained from m by a single swap, violating the second condition of Theorem 4. This shows that at least one state must separate types for identifiability to hold. What is more, this condition is also sufficient, since if at least one state separates types, then Theorem 3 applies.

These matching statements rest on underlying determinant conditions: if state s separates types, then M^s has full rank (it is the identity or a permutation matrix) and $\det M^s \neq 0$, and if state s pools types, then M^s has rank one (two identical columns) and $\det M^s \neq 0$. Global identification on the interior of the simplex means that[9] $\det M(f(a)) \neq 0$ for all $f(a) \in (0, 1)$. Since $\det M(f(a))$ is a linear function of $f(a)$ in the $2 \times 2 \times 2$ case, this is equivalent to: $\det M^a$ and $\det M^b$ have the same sign, and are not both zero. If exactly one state, say a, separates types, then $\det M(f(a)) = f(a) \det M^a$, which is nonzero for all $f(a) \in (0, 1)$. Hence the model is generically (and indeed globally) identifiable.

If both states separate types, two cases arise. When each type chooses identically across states, $M^a = M^b = M$ and $M(f(a))$ is constant in $f(a)$ and invertible, so generic (and indeed global) identification holds. By contrast, when each type chooses differently across states, say M^a is the identity and M^b is a permutation matrix, $\det M(f(a)) = 2f(a) - 1$ which vanishes at a unique interior value of $f(a)$, ensuring generic (though not global) identification. The proof of Theorem 5 generalizes this logic.

The reasoning in Example 7 shows as a corollary of Theorem 4 that in the $2 \times 2 \times 2$ case the parameters are generically identifiable if and only if there exists at least one state that separates types. In subsection 4.2 we prove the result and its generalizations by means of nullspace conditions.

4.1.1 Global identifiability

The characterizing condition for global identifiability closely mirrors that of Theorem 2 in the general framework, with one notable difference. In this context, "global identifiability" is understood with reference to the parameter space in which state distributions are *strictly positive*, $f \in (0, 1)^{|A|}$. Allowing f to assign zero weight to some states would impose the additional requirement that *each* state-specific matrix M^a be invertible. This would trivialize the type–state framework, as it would imply that types are already perfectly separated

[9]Denoting $M(f(a)) = f(a) M^a + (1 - f(a)) M^b$.

within each state. The matching conditions in the following result therefore characterize global identifiability on the interior of the simplex.

Theorem 5. *The parameters in π are globally identifiable on the interior of the simplex if and only if there exists a subset $R \subseteq X$ with $|R| = r$ such that:*

(i) $\mathcal{S}_R \neq \varnothing$;

(ii) For any two state-matchings $(m,\gamma),(m',\gamma') \in \mathcal{S}_R$, m' can be obtained from m via an even number of swaps.

Proof. Please refer to Appendix A.5. □

The $2 \times 2 \times 2$ setting neatly illustrates the kind of different behavioral restrictions that are imposed by generic and global identifiability conditions. The following example is to be contrasted with Example 7:

Example 8. Consider again the type–state framework with two types $T = \{t_1, t_2\}$, two states $A = \{a, b\}$, and two alternatives $X = \{x_1, x_2\}$. Let $f(a) \in (0, 1)$. Note that here necessarily $R = A$ if an R as in the conditions of Theorem 5 exists. We know from Example 7 that type separation in at least one state is necessary for global identifiability (since it is such for generic identifiability).

Suppose *exactly* one state separates types—for instance, $c_{t_1}(a) = x$, $c_{t_2}(a) = y$ and $c_{t_1}(b) = y = c_{t_2}(b)$. Then there is exactly one state-matching, $(m,\gamma) = (xy, aa)$, that uses only one state; and exactly one state-matching, $(m,\gamma') = (xy, ab)$ that uses each state once. The conditions of Theorem 5 are satisfied and there is global identification.

Suppose alternatively that both states separate types. Here there is a crucial distinction, between the case in which *each type chooses in the same way in the two states* (e.g. $c_{t_1}(s) = x$, $c_{t_2}(s) = y$ for $s \in A$), and the case in which *each type swaps choice from one state to the other* (e.g. $c_{t_1}(a) = x = c_{t_2}(b)$, $c_{t_1}(b) = y = c_{t_2}(a)$). It is clear that while in the first case there is a unique matching and thus global identification, in the latter case we can construct multiple matchings differing by one swap, violating condition (ii) of Theorem 5 and unraveling global identification. The determinant analysis of Example 7 underlies this matching analysis.

The upshot of Example 7 is that in the $2 \times 2 \times 2$ case the parameters are globally identifiable if and only if either there is exactly one state that separates types, or both states separate the types but each type chooses identically across states. If each type chooses differently across states, then identification can only be generic.

Finally, we have seen (Theorem 5) that the existence of a state that separates types is sufficient for generic identification. This begs the question: what additional condition ensures global identification? A corollary to Theorem 5 yields the answer. Fix a "reference" state $a^* \in A$ that separates types. Define a binary relation $\rightsquigarrow$ (relative to a^*) on T by

$$t \rightsquigarrow t' \quad \Longleftrightarrow \quad \exists a \in A \text{ such that } c_t(a) = c_{t'}(a^*).$$

This simply means that type t is, in some state, possible at the alternative chosen by type t' in the reference state.

We say that a bijection $\varphi : T \to T$ is a *possible reassignment* (relative to a^*) if

$$\forall t \in T, \quad t \rightsquigarrow \varphi(t).$$

In words: each type t is assigned a "target" type t'. Each type chooses, in some state(s), an alternative chosen by its "target" type in the reference state, with no two types choosing the same alternative (because φ is bijective).

Corollary 1. *Suppose there exists a state $a^* \in A$ that separates types. Then the parameters π are globally identifiable on the interior of the simplex if and only if every possible reassignment φ is an even permutation of T.*

Proof. Please refer to Appendix A.5. □

4.2 Identification of the type distribution via nullspace conditions

The previous analysis provides powerful and general characterizations of generic identification. However, those conditions are not completely informative on how different states interact to produce or destroy identification; and on the precise behavioral implications of such failures. This section complements that analysis by adopting a geometric, linear-algebraic perspective rather than a combinatorial one. We derive nullspace conditions that make explicit the directions in aggregate choice space along which identification fails, and we interpret lack of identification as the existence of nontrivial directions in aggregate choice space that can be generated in multiple ways.

This is transparently translated to behavioral conditions. The key conceptual step is the notion of "typical splits", introduced in Definition 6 below. Identification fails precisely when distinct subsets of types generate the same aggregate choice splits—that is, the same differences in choice frequencies between two alternatives—so that these directions cannot be uniquely attributed to individual types.

For brevity and to obtain clear conditions, we focus on the case of two states, and on generic identification only.

4.2.1 The 2×2 and 3×3 cases

We proceed in stages, picking up from the simple case of two states, two types, and two alternatives from the end of the previous section. In this case, Equation (2) reduces to the convex combination of two 2×2 matrices, where generic identification can be characterized as follows.

Theorem 6. *For $p = M\pi$, where $M = \sum_{a \in A} f(a)M^a$ is a 2×2 matrix as defined in Equation* (2) *with $A = \{a, b\}$ and $T = \{t_1, t_2\}$, the following statements are equivalent*

(i) The parameters in π are generically identifiable

(ii) $\text{Null}(M^a) \cap \text{Null}(M^b) = \{\mathbf{0}\}$

(iii) $c_{t_1}(a) \neq c_{t_2}(a)$ or $c_{t_1}(b) \neq c_{t_2}(b)$

Proof. Please refer to Appendix A.7. □

This first result via nullspace conditions establishes that the parameters in π are generically identifiable if and only if there is a state that separates types (condition (iii) of Theorem 6). On the other hand, the theorem also provides a purely algebraic characterization of generic identification. The equivalence of Condition (i) and (ii) shows that the parameters in π are generically identifiable if and only if the intersection of the nullspaces of M^a and M^b is the trivial one, i.e., the only $\mathbf{x}$ satisfying both $M^a\mathbf{x} = \mathbf{0}$ and $M^b\mathbf{x} = \mathbf{0}$ is the zero-vector. Given the specific one-hot structure of these matrices, this nullspace condition is equivalent to restricting one of the individual nullspaces to the trivial one, i.e., $\text{Null}(M^a) = \{\mathbf{0}\}$ or $\text{Null}(M^b) = \{\mathbf{0}\}$. This demonstrates the equivalence of Condition (ii) and (iii).

Next, we turn to the case of two states, three types, and three alternatives. As in the previous case, it remains necessary for identification that not all states pool types. But, this condition alone is *not* sufficient for generic identification in the 3×3 case. The next example illustrates why.

Example 9. This example consists of two parts. This first part motivates why at least one state must separate types for identifiability when there are three alternatives and three types. Let $A = \{a, b\}$ be the state space, $T = \{t_1, t_2, t_3\}$ the type space, and $p = M\pi$, where $M = f(a)M^a + f(b)M^b$, for all $f(a), f(b) \in [0, 1]$, $f(a) + f(b) = 1$, is a 3×3 matrix. Now suppose that both states pool types, i.e., let $c_{t_1}(a) = c_{t_2}(a)$ and $c_{t_1}(b) = c_{t_2}(b)$. For instance, let M be given as follows

$$f(a)\underbrace{\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}}_{M^a} + f(b)\underbrace{\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{M^b} = \underbrace{\begin{bmatrix}f(a) & f(a) & 0\\ f(b) & f(b) & f(a)\\ 0 & 0 & f(b)\end{bmatrix}}_{M}$$

Then, the parameters in π are not identifiable. To see this observe that M is clearly not invertible, because $\mathrm{Null}(M) = \mathrm{Null}(M^a) \cap \mathrm{Null}(M^b) \neq \{\mathbf{0}\}$.

The next part shows that the existence of a type separating state does not by itself guarantee generic identification. To see this, let the matrix M be given as follows

$$f(a)\underbrace{\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}}_{M^a} + f(b)\underbrace{\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{M^b} = \underbrace{\begin{bmatrix}f(a) & 1 & f(b)\\ f(b) & 0 & f(a)\\ 0 & 0 & 0\end{bmatrix}}_{M}$$

Note that no two columns of M coincide for all $f(a)$ and $f(b)$. In other words, for any two types, there exists a state that separates them. Nevertheless, the parameters in π are not identifiable. Indeed, M is not invertible, because it has a zero row indicating that the corresponding alternative is never chosen. What captures this algebraically is the fact that the nullspaces of the *transposes* of M^a and M^b have the following non-trivial intersection,

$$\mathrm{Null}(M^T) = \mathrm{Null}(M^{a^T}) \cap \mathrm{Null}(M^{b^T}) = \mathrm{span}\left\{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right\}.$$

Hence, for generic identification in the 3×3 case some additional condition is needed besides not all states pooling types. The additional requirement is that *aggregate choices spread over all three alternatives*. The next result shows that, taken together, these two conditions characterize generic identification in the 3×3 case.

Theorem 7. *For $p = M\pi$, where $M = \sum_{a \in A} f(a)M^a$ is a 3×3 matrix as defined in Equation* (2) *with $A = \{a, b\}$, $X = \{x, y, z\}$, and $T = \{t_1, t_2, t_3\}$, the following are equivalent*

(i) The parameters in π are generically identifiable

(ii) $\text{Null}(M^a) \cap \text{Null}(M^b) = \{\mathbf{0}\}$ and $\text{Null}(M^{a^T}) \cap \text{Null}(M^{b^T}) = \{\mathbf{0}\}$

(iii) For all $t, t' \in T$ with $t \neq t'$, there exists an $a \in A$ such that $c_t(a) \neq c_{t'}(a)$; and, for all $w \in X$, there exist $a \in A$ and $t \in T$ such that $c_t(a) = w$

Proof. Please refer to Appendix A.8. □

The equivalence between Condition (i) and (iii) establishes that, in the 3×3 case, generic identification is equivalent to two requirements. The first is the existence of a type separating state. The second is that none of the three alternatives is never chosen. The theorem also shows how this translates in terms of nullspaces. The equivalence between Condition (i) and (ii) shows that the parameters in π are generically identifiable if and only if *both* the intersection of the nullspaces of M^a and M^b and the intersection of the nullspaces of M^{a^T} and M^{b^T} are the trivial ones. Given the one-hot structure of these matrices, $\text{Null}(M^a) \cap \text{Null}(M^b) = \{\mathbf{0}\}$ is equivalent to the existence of a type separating state, and $\text{Null}(M^{a^T}) \cap \text{Null}(M^{b^T}) = \{\mathbf{0}\}$ is equivalent to none of the three alternatives being never chosen.

With three alternatives and three types, the aggregate choice shares may allow to identify all types, *even though choice shares from any single state alone may not do so*. Neither the data in M^a alone, nor the data in M^b alone would suffice for identifying the parameters in π, but together they guarantee generic identification. The next example illustrates such a situation.

Example 10. Let $A = \{a, b\}$ be the state space and let there be three types and three alternatives, i.e., $|T| = |X| = 3$. Consider $p = M\pi$, where the 3×3 matrix $M = f(a)M^a + f(b)M^b$, for all $f(a), f(b) \in [0,1]$, $f(a) + f(b) = 1$, is given as follows

$$f(a)\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{M^a} + f(b)\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{M^b} = \underbrace{\begin{bmatrix} f(a) & f(a) & 0 \\ f(b) & 0 & f(a) \\ 0 & f(b) & f(b) \end{bmatrix}}_{M}$$

Neither the data in M^a alone nor that in M^b alone suffices for identification, as each such matrix has a zero row. Taken together, however, this data allows for generic identification. For any two types, there exists a state that separates them. Furthermore, M has no zero row which shows that aggregate choices spread over all three alternatives. In other words, neither M^a nor M^b are invertible. But, since $\text{Null}(M^a) \cap \text{Null}(M^b) = \{\mathbf{0}\}$ and $\text{Null}(M^{a^T}) \cap \text{Null}(M^{b^T}) = \{\mathbf{0}\}$, Theorem 7 implies that their convex combination M leads to π being generically identifiable from $p = M\pi$. Indeed, since $\det(M) = -f(a)f(b)(f(a)+f(b))$, the parameters in π are generically identifiable as long as neither $f(a)$ nor $f(b)$ are zero.

4.2.2 The general $r \times r$ case

The previous results suggest that a single type separating state may suffice for generic identification. Indeed, in line with Theorem 3, this is true in the most general $r \times r$ case. That is, it holds for any $r \in \mathbb{N}$ and any finite number of $r \times r$ matrices. But the result can be further strengthened. For the parameters in π to be generically identifiable from $p = M\pi$ and $M = \sum_{a \in A} M^a$, it suffices that there exists some subset $B \subseteq A$ such that the parameters in π would be generically identifiable from $p = N\pi$ and $N = \sum_{b \in B} f(b)M^b$. The next result spells out this fact.

Theorem 8. *If the parameters in π are generically identifiable from $p = N\pi$ where $N = \sum_{b \in B} f(b)M^b$ is as defined in Equation* (2)*, then they continue to be generically identifiable from $p = M\pi$, where $M = \alpha M^a + (1-\alpha)N$.*

Proof. Please refer to Appendix A.9. □

The implication of this result for identification is the following: if data from some subset of states suffices for the parameters in π to be generically identifiable, then the inclusion of additional states cannot undo this identification. A simple corollary is that, within the type–state framework, the parameters in π are generically identifiable if and only if there exists at least a parameter configuration that is unambiguously revealed by the data. The next result formalizes this observation.

Corollary 2. *The parameters in π are generically identifiable from $p = M\pi$, where $M = \sum_{a \in A} f(a)M^a$ is as defined in Equation* (2)*, if and only if there exists some probability measure $\overline{f}$ on A such that $p = \overline{M}\pi$ and $p \neq \overline{M}\pi'$ for all $\pi' \neq \pi$, where $\overline{M} = \sum \overline{f}(a)M^a$.*

Proof. Please refer to Appendix A.10. □

This result implies in particular that if the parameters in π are *not* generically identifiable, then there does not exist *any* parameter configuration that is uniquely revealed by the data p. While lack of generic identification is defined as the failure of identification on a set of parameter values of nonzero Lebesgue measure, Corollary 2 implies a stronger conclusion, already encountered in the matching approach: *when generic identification of π fails, it fails structurally.*

Finally, we extend the conditions from Theorem 6 and Theorem 7 to the general $r \times r$ case. The ensuing theorem specifies the form of qualitative behavioral heterogeneity that is both necessary and sufficient for generic identification. To state it precisely, we first introduce a key definition, formalizing when a given pattern of aggregate choice shares can be regarded as typical of a group of types.

Definition 6. Given M as defined in Equation (2) and some non-empty set of types $S \subseteq T$, the split between two alternatives $x, y \in X$ is *typical of S* if

$$(\mathbf{e}^x - \mathbf{e}^y) \in \text{span}\{M_{*j} : \text{column } j \text{ corresponds to some type } t \in S\}$$

where for $w \in \{x, y\}$, $\mathbf{e}^w$ is defined as follows: for all $i = 1, \ldots, r$

$$\mathbf{e}_i^w = \begin{cases} 1, & \text{if Row } i \text{ corresponds to alternative } w \\ 0, & \text{otherwise} \end{cases}$$

Intuitively, a split between two alternatives is typical of a set of types if the aggregate difference in their choice frequencies between these alternatives can be replicated as a linear combination of the choice patterns generated by those types alone. In this case, observed differences in frequency between the two alternatives do not provide information that helps distinguish a particular type within the set, as the same aggregate split could be generated by reallocating choice probabilities among the remaining types. Typical splits therefore capture directions in aggregate choice space that are not informative about the behavior of individual types, and play a central role in determining when different patterns of behavior can or cannot be uniquely attributed to distinct types.

The next example illustrates the idea of a split between two alternatives being typical of some set of types.

Example 11. Let $A = \{a, b\}$ be the state space and let there be four types, $T = \{1, 2, 3, 4\}$ and four alternatives, $X = \{w, x, y, z\}$. Further, let the 4×4 matrix $M = f(a)M^a + f(b)M^b$, for all $f(a), f(b) \in [0,1]$, $f(a)+f(b)=1$, be given as follows

$$f(a)\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{M^a} + f(b)\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{M^b} = \underbrace{\begin{bmatrix} f(a) & f(a) & 0 & 0 \\ f(b) & 0 & f(a) & 0 \\ 0 & f(b) & 0 & f(a) \\ 0 & 0 & f(b) & f(b) \end{bmatrix}}_{M}$$

where type $t \in T$ corresponds to the t-th Column of M, and alternative w to Row 1, x to Row 2, y to Row 3, and z to Row 4. Note that $\det(M) = 0$. Now, for $f(a), f(b) \neq 0$, the split between alternatives x, y is typical of the set of types $\{3, 4\}$, as

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{f(a)}(M_{*3} - M_{*4})$$

But this same split is also typical of the set of types $\{1, 2\}$. To see this, observe that $f(a)(M_{*1} - M_{*2}) = f(b)(M_{*3} - M_{*4})$.

The definition of a split between two alternatives being typical of a set of types provides the behavioral notion of heterogeneity that is both necessary and sufficient for identification. In the previous example, the fact that $\det(M) = 0$ indicates that the same split cannot be typical for distinct sets of types. The next theorem formalizes this observation and establishes an equivalent algebraic condition, expressed in terms of non-trivial intersections between the nullspaces of the relevant matrices.

Theorem 9. *For an $r \times r$ matrix $M = \sum_{a \in A} f(a) M^a$ as defined in Equation* (2) *with $A = \{a, b\}$, $|X| = |T| = r$, the following are equivalent*

(i) The parameters in π are generically identifiable

(ii) For almost all $\{f(a)\}_{a \in A}$, for all non-trivial $\mathbf{x} \in \text{Null}(M^a)$, all non-trivial $\mathbf{y} \in \text{Null}(M^b)$, and all $\mathbf{z} \in \text{span}\{\mathbf{x}, \mathbf{y}\}^{\perp}$, it holds that

$$(f(a)M^a + f(b)M^b)\mathbf{z} \neq M^a\mathbf{y} + M^b\mathbf{x}$$

(iii) There exists $a, b \in A$, $a \neq b$, such that, for all $t, t' \in T$, $t \neq t'$, with $c_t(a) = c_{t'}(a)$, it holds that $c_t(b) \neq c_{t'}(b)$ and the split between $c_t(b)$ and $c_{t'}(b)$ is not typical of $(T \setminus \{t, t'\})$

Proof. Please refer to Appendix A.11. □

Condition (iii) conveys two key implications. First, for any two types, there must exist at least one state that separates them. Second, if two types are pooled by one state, then the way in which their choices differ in another state cannot itself be typical of the choices of all other types. The condition connects directly to the identification-by-matching results from the previous section. If two types are pooled every state, they cannot be matched to distinct alternatives. Moreover, if two types are pooled by one state and are separated by another, but their difference in choice is also typical of another subset of types, then the uniqueness requirement of the matching result in Theorem 4 fails to hold.

To understand Condition (ii), first observe that Theorem 8 implies that the parameters in π are generically identifiable whenever there is no non-trivial $\mathbf{x} \in \text{Null}(M^a)$ or no non-trivial $\mathbf{y} \in \text{Null}(M^b)$. It follows that any non-trivial $\mathbf{w} \in \text{Null}(M)$ can be written as a linear combination involving some such non-trivial $\mathbf{x}$ and $\mathbf{y}$. That is, any such non-trivial $\mathbf{w}$ can be decomposed as follows

$$\mathbf{w} = \mathbf{x} + \mathbf{y} + \mathbf{z} \tag{3}$$

for some non-trivial $\mathbf{x} \in \text{Null}(M^a)$, some non-trivial $\mathbf{y} \in \text{Null}(M^b)$ and some $\mathbf{z} \in \text{span}\{\mathbf{x}, \mathbf{y}\}^{\perp}$. Now, the parameters in π being generically identifiable implies that, for almost all $\{f(a)\}_{a \in A}$, there exists no such non-trivial $\mathbf{w} \in \text{Null}(f(a)M^a + f(b)M^b)$. In other words, for almost all $\{f(a)\}_{a \in A}$, all $\mathbf{w}$ from Equation (3) satisfy $(f(a)M^a + f(b)M^b)\mathbf{w} \neq \mathbf{0}$. As the proof of Theorem 9 establishes this inequality is equivalent to the one in Condition (ii).

Condition (ii) furthermore generalizes the conditions of the previous identification theorems in this section. First, observe that Condition (ii) generalizes Theorem 8 in the following way: If π is globally identifiable from $p = M^a\pi$ (or $p = M^b\pi$), the nullspace of M^a (or M^b) is the trivial one, i.e., it only contains the zero vector. Since Condition (ii) only refers to non-trivial elements of the nullspaces of M^a and M^b, in line with Theorem 8, Condition (ii) is then satisfied automatically. Next, observe that $\text{span}\{\mathbf{x}, \mathbf{y}\}^{\perp}$ always contains the zero vector. So, Condition (ii) rules out that the nullspaces of M^a and M^b intersect in a non-trivial way. To see this, suppose otherwise and let there be some $\mathbf{x} \neq \mathbf{0}$ which is an element of both

$\mathrm{Null}(M^a)$ and $\mathrm{Null}(M^b)$. But, then $\mathbf{0} = f(a)M^a\mathbf{y} + f(b)M^b\mathbf{x}$, contradicting Condition (ii), as $\mathbf{0} \in \mathrm{span}\{\mathbf{x}, \mathbf{y}\}^{\perp}$. As such, Condition (ii) generalizes the corresponding condition in Theorem 6 from the 2×2 case. Since $\mathrm{span}\{\mathbf{x}, \mathbf{y}\}^{\perp}$ contains the zero vector, another way in which Condition (ii) may fail is when $M^a\mathbf{y} = M^b\mathbf{x}$ holds, for some non-trivial $\mathbf{x} \in \mathrm{Null}(M^a)$ and some non-trivial $\mathbf{y} \in \mathrm{Null}(M^b)$. Therefore, Condition (ii) also generalizes the corresponding condition in Theorem 7 from the 3×3 case.[10]

Remark 2. All characterization results in this section can be generalized to the case of an $n \times r$ matrix M, where $n \geq r$. To this end, the corresponding statements need to be adjusted to saying that the matrix M contains an $r \times r$ submatrix to which these statements then apply.

5 Multiple choice occasions and the identification of type-level choice probabilities

This section advances the analysis by studying how identification can be extended to type-level choice probabilities. To this end, we consider an analyst who observes choice data from a finite number of occasions $j \in \{1, \ldots, J\}$. The extreme case of $J = 1$ is that of a one-shot observation, which was the focus of the previous sections. We assume that the distribution π of types remains constant across occasions. For each occasion j, agents choose from a feasible set X_j. We assume that choices are independent across occasions conditional on the type. The notation and definitions introduced earlier extend in the natural way to all j-indexed objects of this section.

The data set is now described as a tensor[11] S of order J with dimensions $n_1 \times \ldots \times n_J$. A typical entry $S_{k_1,\ldots,k_J}$ is the share of the population choosing alternative x_{k_j} from X_j for $j \in \{1, \ldots, J\}$. Hence, S contains the joint distribution of choices for the J occasions. We can express S using M_j and π as

$$S = \sum_{h=1}^{r} \pi(h) \bigotimes_{j=1}^{J} M_j \mathbf{1}_h, \tag{4}$$

[10] We should also note that, although parts of the proof make use of linear decompositions of vectors into components lying in different subspaces, no inner-product structure or notion of orthogonality plays any *substantive* role in the argument. All results depend only the one-hot structure of the matrices M^a and M^b, and on the resulting row-sum identities that characterize typical splits. Any reference to complementary subspaces is purely a notational convenience (and could in fact be replaced by an arbitrary linear complement). In particular, the identification results are invariant to the choice of inner product and do not rely on any metric properties of $\mathbb{R}^r$. The characterization remains in this sense entirely qualitative in nature.

[11] A tensor is a multidimensional array that extends the concept of a matrix to an arbitrary number of indices, known as the tensor's order. Its dimensions specify how many values each index can take, generalizing the numbers of rows and columns in a matrix.

where $\otimes$ is the outer product operator and $\mathbf{1}_h$ is the unit vector for component h.[12]

It turns out that $J = 3$ is sufficient for the generic identification of the model in Equation (4), provided the possibility relations are sufficiently rich. Under this condition, one can apply algebraic results on tensor decompositions to identify the model. The uniqueness properties of these decompositions have been extensively studied, beginning with Kruskal's [25] fundamental theorem and later refined by Allman, Matias, and Rhodes [1] and others. Because the spirit of our analysis is that of identifying the parameters from minimal data, we restrict the agent's choices to the three occasions required. Additional occasions neither aid nor impede identification in this setting, whereas two occasions are insufficient.

5.1 General framework

In the general case, the parameter space of the model is a full dimensional subset of $[0,1]^m$ where $m = (r-1) + \sum_{j=1}^{3}(|\wedge_j| - r)$. The model's parameterization map takes values in $[0,1]^{n_1 \times n_2 \times n_3}$.

The next result gives a sufficient condition for generic identification.

Theorem 10. *Suppose that there exist natural numbers $v_j \le r$ for $j \in \{1,2,3\}$ such that:*

(i) $v_1 + v_2 + v_3 \ge 2r + 2$.

(ii) For any $j \in \{1,2,3\}$ and any v_j types in T, there is a matching of these types with alternatives in X_j.

Then the parameters in π, M_1, M_2, and M_3 are generically identifiable, up to label swapping.

Proof. Please refer to Appendix A.12. □

Note that for the application of Theorem 10 we need at least $2r + 2$ alternatives across the three occasions that have positive choice shares. Additionally, because identification is unique up to a simultaneous permutation of the entries of π and columns of M_1, M_2, and M_3, we do not know a priori which of the inferred values of π corresponds to a particular type. However, the correct labeling is often revealed by the pattern of 0's in M_j. The matrix in Example 1 may serve to illustrate this.

[12]Thus, $M_j \mathbf{1}_h$ is the hth column of M_j. The outer product operation then creates a rank-1 tensor of order J. A tensor is said to be of rank 1 if it is an outer product of vectors. Therefore, S is a convex combination of r rank-1 tensors.

Example 12. Consider the following matrix form, with rows corresponding to four alternatives x, y, z, and w, and columns corresponding to types t_1, t_2, and t_3,

$$M_j = \begin{array}{cc} & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{array}{c} x \\ y \\ z \\ w \end{array} & \begin{pmatrix} \textcolor{red}{\bullet} & \bullet & \bullet \\ 0 & 0 & \textcolor{red}{\bullet} \\ \bullet & \textcolor{red}{\bullet} & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \end{array}.$$

Note that $t \wedge_j x$ and $t \wedge_j z$ for all types t, but $t \wedge_j y$ and $t \wedge_j w$ only for $t = t_3$. It is easy to see that the second condition of Theorem 10 is satisfied with $v_j = 3$ (for example, take the matching represented by the red ($\textcolor{red}{\bullet}$) dots).

Clearly, when a type-to-alternative matching as in Definition 2 exists, it implies that $v_j = r$, making it easier to satisfy the first condition of Theorem 10. The next examples illustrates a situation where less information is conveyed by choices.

Example 13. Consider the following matrix form:

$$M_j = \begin{array}{cc} & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \end{array}.$$

In this case, the second condition is only true for $v_j = 1$ because there is no matching covering both types t_1 and t_2. Intuitively, this pattern of choices does not help to identify the shares of t_1 and t_2 because the agents of these types always choose x.

5.2 Type-state framework

We now consider multiple occasions of choice in the type-state framework. The states are allowed to vary with the occasions, with A_j denoting the set of states at occasion j. When the conditions of Theorem 4 are satisfied for each occasion, we can prove the generic identifiability of the parameters of our model.

Theorem 11. *Consider the model described by Equations* (2) *and* (4) *with parameters π and f_j. If, for any occasion $j \in \{1, 2, 3\}$, the conditions of Theorem 4 hold, then π and $\{M_j : j \in \{1, 2, 3\}\}$ are generically identified, up to label swapping.*

Proof. Please refer to Appendix A.13. □

Theorem 11 provides a sufficient condition for the generic identification of π and M_j, $j \in \{1,2,3\}$. It is sometimes also possible to recover f_j from Equation (2). In particular, this occurs when there exists a type that selects a different alternative in every state $a \in A_j$. More generally, Equation (2) defines a system of linear equations in the unknowns $\{f(a)\}_{a \in A_j}$, which may or may not admit a unique solution.

Theorems 4 and 11 can be generalized to the case where the state distribution f_j depends on the type. In that case, instead of Equation (2) we would have that each column of M_j is a different convex combination of the corresponding columns of $\{M^a : a \in A_j\}$.

6 Applications

In this section we illustrate concretely how our framework can be applied, by studying specific environments.

6.1 Choice over vectors of characteristics

Suppose that consumer goods are described in terms of measurable characteristics (after Lancaster [27]), or *features*. Denote by $F = \{1, \ldots, N\}$ the set of possible features.[13] A good $x = (x_1, ..., x_N)$ is described by the amounts x_i, for all features $i \in F$, of that good (while we use the term "good" for simplicity, as explained below we allow features to be undesirable).

Consumers choose from feasible set $Y \subseteq \mathbb{R}^N$. We assume that the feasible set Y is a nonempty closed bounded convex set of $\mathbb{R}^N$ with the boundary denoted by ∂Y.

A *type* here is an $I \subseteq F$ corresponding to the nonempty set of features that the consumer considers relevant for the description of the alternatives. A consumer type I chooses an $x^* \in Y$ that is "optimal" with respect to the features I. The next definition specifies the notion of optimality.

Definition 7. Given an evaluation function $e : F \to \{-1, 0, 1\}$, a point $x \in Y$ is *e-admissible* if $y \notin Y$ whenever $e(i)\, y_i \geq e(i)\, x_i$ for all $i \in F$ and $e(i^*)\, y_{i^*} > e(i^*)\, x_{i^*}$ for some $i^* \in F$.

The consumer's evaluation function e indicates if the consumer values a feature positively, negatively, or ignores it. Clearly, for a consumer of type I, we impose that $e(i) \neq 0$ if and only if $i \in I$. The behavioral assumption is that for each consumer there exists an (unobserved) evaluation function e such that the choice $x^* \in Y$ is e-admissible. This requirement

[13] [24] studies a version of this model using only *deterministic* choices.

is similar to Pareto optimality; however, it is applied only to the dimensions in I, and the directions of monotonicity specified by e are not a priori known to the analyst.

Definition 8. A type $I \subseteq F$ is *possible* at $x^* \in Y$ if there exists an evaluation function e such that $e(i) \neq 0 \iff i \in I$ and x^* is e-admissible.

Let $X \subset Y$ be a finite set of goods with positive choice shares. For each type $I \subseteq F$ and each good $x \in X$, we write $I \wedge x$ if type I is possible at $x \in Y$ according to Definition 8.

As an example, let $N = 2$, so the types are $\{1\}$, $\{2\}$, and $\{1, 2\}$. The feasible set Y is shown in Figure 1. As one can readily verify, the possible types at each goods $X = \{x, y, z, w\}$ coincide with those described by M_j in Example 12. Hence, Theorem 1 guarantees that the distribution of the types $\{1\}$, $\{2\}$, and $\{1, 2\}$ can be generically recovered from the observed choices. With three occasions, Theorem 10 further shows that we can generically identify both the type distribution and the matrices M_1, M_2, and M_3.

The type-state specialization of the general framework can be applied to a population of consumer types that satisfy a linearity restriction. Formally, let us say that a type $I \subseteq F$ is *linear* at $x^* \in Y$ if there exists an $a \in \mathbb{R}^N$ such that $a_i \neq 0 \iff i \in I$ and $x^* \in \operatorname{argmax}_{x \in Y} \langle a, x \rangle$. It is easy to show that if a type is linear at x^*, then it is also possible at x^*.

For this application, we assume that the feature coefficients are drawn from a finite set of values. Thus, f_j is interpreted as a probability measure defined on a finite set $A_j \subset \mathbb{R}^N$ of feature values vectors. After a realization of a according to f_j, the consumer solves

$$\max_{x \in Y_j} \langle \kappa_I(a), x \rangle \tag{5}$$

where $\kappa_I : \mathbb{R}^N \to \mathbb{R}^N$ is defined as $\kappa_I(a)_i = a_i$ if $i \in I$ and $\kappa_I(a)_i = 0$ otherwise.

We exclude the non-generic case when the consumer problem (5) does not have a unique solution.[14] Thus, Equation (5) determines a choice function in the sense previously specified, and the results of Section 4 can be applied to ensure that the distribution of consumer types can be identified.

Some additional identification analysis can be carried out in the model: in fact, M_j contains some information about the evaluation functions in the consumer population. Let us consider the example of Figure 1 again. Recall that each consumer's choice is e-admissible with respect to the consumer's evaluation function e. The first column of M_j describes the tastes of the consumers for whom only feature x_1 matters. Precisely, $M_j(1, 1)$ is the share of $e(1) = 1$ (x_1 is desirable) and $M_j(3, 1)$ is the share of $e(1) = -1$ (x_1 is undesirable). Likewise,

[14] Alternatively, a tie-breaker assumption can be used.

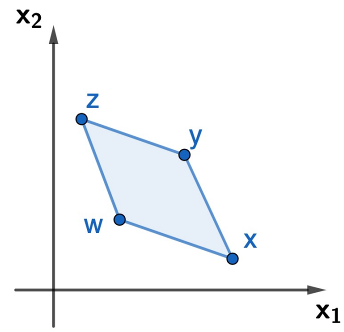

Figure 1: Choice problem in Example 12

the second column describes the tastes of the type $\{2\}$ agents. Finally, the third column describes the tastes of the type $\{1,2\}$ agents. For example, the consumers choosing y (w) must evaluate both features positively (resp., negatively), while this is less certain for those choosing x or z. If the population consists of only linear types, then the third column of M_j shows the shares of the type $\{1,2\}$ consumers whose vectors of feature values belong to the normal cones at points x, y, z, and w.

6.2 Incomplete preferences

Suppose that the set $X = \{x_1, \ldots, x_n\}$ of alternatives is linearly ordered from best to worst according to a strict preference relation $\succ$, i.e., $x_1 \succ x_2 \succ \cdots \succ x_n$. Agents' preferences are *incomplete* in the sense that alternatives that are sufficiently close in this ordering are treated as incomparable or indifferent.[15] Formally, an agent of type t_l can distinguish between two alternatives only if they are at least l positions apart in the ordering induced by $\succ$. Let $\succ_l$ denote the preference relation of type t_l. It follows that each $\succ_l$ is a coarsening of $\succ$: $\succ_1 = \succ$, $\succ_{l+1} \subset \succ_l$ for all l, and $\succ_n$ is the empty relation. The set of types is given by $T = \{t_1, \ldots t_n\}$.

Each type never chooses a dominated alternative and randomizes among undominated ones. In this sense, t_1 represents the fully rational type, whereas t_n corresponds to the fully random type. Observe that $t_l \wedge x_k$ if and only if $l \geq k$. Consequently, the parameters populate an upper-triangular type-conditional choice matrix M. This structure guarantees the existence of a unique matching, and the parameters in π are globally identifiable from the observed choice probabilities $p = M\pi$ by Theorem 1. Moreover, if data from multiple choice occasions are available, the analyst can also identify the entries of M (the randomization parameters). In particular, in the three-occasion setting discussed in Section 5, we have

[15] Incomplete preferences have been studied extensively in decision theory and social choice, both as a primitive departure from classical rationality (Aumann [4], Dubra *et al.* [16]) and as an outcome of bounded rationality or limited discrimination (Masatlioglu *et al.* [33], Manzini and Mariotti [30]).

$v_1 = v_2 = v_3 = n$ and $r = n$. Hence, both conditions of Theorem 10 are satisfied, implying that the parameters in π, M_1, M_2, and M_3 are generically identifiable.

Example 14. Let $X = \{x, y, z, w\}$ and $x \succ y \succ z \succ w$. The induced preference relations for the different types are given by

$$\succ_1 = \succ, \ \{x \succ_2 z, x \succ_2 w, y \succ_2 w\}, \ \{x \succ_3 w\}, \ \succ_4 = \emptyset.$$

The corresponding sets of undominated alternatives are therefore $\{x\}$ for type t_1, $\{x, y\}$ for type t_2, the top three alternatives for type t_3, and all four alternatives for type t_4. As a result, the type-conditional choice matrix M takes the form

$$M = \begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \!\!\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{pmatrix} \end{array}.$$

This matrix clearly admits a unique matching, implying that the type distribution is globally identifiable. Moreover, if data from multiple choice occasions are available, the choice matrix M itself is also generically identifiable.

The type-state specification of the framework is useful when the exact nature of the agents' randomization behavior is known. To illustrate, suppose that agents resolve incompatibilities or indifferences under the influence of the salience of the alternatives. Formally, let each state $a \in A$ specify a linear salience ordering $\rhd_a$ over the set X of alternatives. An agent of type t_l in state a chooses the maximizer of $\rhd_a$ over the agent's set $\{x_1, \ldots, x_l\}$ of undominated alternatives. Each state thus generates a type-conditional choice matrix M^a with one-hot columns. The overall type-conditional choice matrix is obtained by aggregation, $M = \sum_{a \in A} M^a$. According to Theorem 3, the type distribution π is generically identifiable from $p = M\pi$ if the type is fully identified in at least one state $a^* \in A$.

One case when the type is fully identified given $a^* \in A$ arises when the salience ordering $\rhd_{a^*}$ is the *reverse* of the preference ordering $\succ$. In this case, each type t_l selects x_l, i.e., $c_{t_l}(a^*) = x_l$ for all l, and the corresponding choice matrix M^{a^*} is the identity matrix. In fact, the presence of the salience ordering that is opposite to $\succ$ is not only sufficient but also necessary for identification. Indeed, such an ordering is the only one under which the worst alternative x_n can be matched to a type. Hence, it follows from Theorem 4 that the parameters in π are generically identifiable if and only if there exists some $a^* \in A$ such that $\rhd_{a^*}$ is the reverse of $\succ$.

As an illustration, consider Example 14 with two states $A = \{a, b\}$, where the salience ordering $\rhd_a$ coincides with the preference ordering $\succ$, while $\rhd_b$ is the reverse of $\succ$. Let $f(a)$ and $f(b)$ denote the probabilities of states a and b, respectively. The equation $f(a)M^a + f(b)M^b = M$ then takes the form

$$f(a)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + f(b)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} f(a)+f(b) & f(a) & f(a) & f(a) \\ 0 & f(b) & 0 & 0 \\ 0 & 0 & f(b) & 0 \\ 0 & 0 & 0 & f(b) \end{bmatrix}.$$

For a generic choice of f, the resulting matrix M is invertible, implying that the distribution π is globally identifiable.

7 Concluding remarks

We provide necessary and sufficient conditions under which aggregate choice shares from a single set of alternatives guarantee identifiability (generic or global) of the type distribution, as well as sufficient conditions for generic identification of both the type distribution and the type-level choice probabilities. A central message of our analysis is that the informational content of aggregate choice data depends less on its volume per se than on the structure of the qualitative behavioral patterns it embeds. Strong identification results can be obtained without imposing parametric structure on decision rules, and without requiring rich menu variation or many repeated observations. At the same time, our results clarify the limits of what can be learned from aggregate data: when behavioral heterogeneity is insufficient, even generic identification breaks down, regardless of the amount of data available.

While the broad link between diversity in behvior and identifiability is intuitive, a contribution of our analysis is to make the required notion of "qualitative behavioral heterogeneity" precise. We do so by characterizing it combinatorially, through the notion of "matchings" between types and alternatives, and by providing an equivalent operationalization in algebraic terms, via interpretable nullspace conditions on the relevant matrices.

By working with abstract behavioral types rather than more specific primitives, the framework developed here is designed to apply to a wide range of behavioral theories—such as limited consideration, salience-based choice, status quo bias, or incomplete preferences—as well as to more ad hoc behavioral hypotheses tailored to particular contexts. Moreover, the qualitative nature of the characterizing properties frees the researcher from the need for precise knowledge of type-level choice probabilities, even at a merely parametric level.

We have focused on identification under extremely undemanding informational assumptions. In many empirical settings, the analyst will have access to additional information, such as observations from multiple menus, covariates, or partial knowledge of the cognitive processes underlying choice. An important direction for future research is to extend the framework developed here to incorporate such additional sources of information and to assess the extent to which they further strengthen identification.

A Proofs

A.1 Proof of Theorem 1

Proof. Note that generic identifiability of π here is equivalent to the columns of M being generically independent.

If the columns of M are generically independent, then there exists an $r \times r$ submatrix M' of M such that $\det(M')$ is not identically 0. Note that $\det(M')$ is a multivariate polynomial in the parameters $\{M'(k,l) : t_l \wedge x_k\}$. From the Leibniz determinant formula

$$\det(M') = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \prod_{l=1}^{r} M'(\sigma(l), l),$$

(where S_r denotes the set of permutations on $\{1, ..., r\}$) it follows that there exists a permutation σ^* of the rows of M' such that the monomial

$$\prod_{l=1}^{r} M'(\sigma^*(l), l)$$

is not identically 0. This is possible only if all of the variables in the monomial are not identically 0. Hence, $t_l \wedge x_{\sigma^*(l)}$ for all $l \in \{1, \ldots, r\}$ and there exists a matching m defined as $m(t_l) = x_{\sigma^*(l)}$. This proves one direction of the assertion on generic identifiability in the statement.

Conversely, if there exists a matching m, then we can define a permutation σ^* for all $l \in \{1, \ldots, r\}$ as the unique permutation satisfying equation $m(t_l) = x_{\sigma^*(l)}$. Then, we have $t_l \wedge x_{\sigma^*(l)}$ for all $l \in \{1, \ldots, r\}$. Form an $r \times r$-submatrix M' of M using the rows corresponding to the alternatives in the image of m. Choosing the point in the parameter space defined by $M'(k,l) = 1$ if $k = \sigma^*(l)$ and $M'(k,l) = 0$ otherwise, we have $\det(M') = 1$. Hence, $\det(M')$ is not identically 0, which means that the r columns of M are generically independent. This concludes the proof of the other direction.

When the matching m is unique, for all $\sigma \in S_r$ with $\sigma \neq \sigma^*$ we have $M'(\sigma(l), l) = 0$ for some l (otherwise we would have found another matching different from m), whereas $\prod_{l=1}^{r} M'(\sigma^*(l), l) \neq 0$ since $M'(\sigma^*(l), l) > 0$ for all l. Equivalently, in the Leibniz expansion there is exactly one term that is nonzero, and we conclude that $\det(M') \neq 0$ for all parameter values. This shows that the columns of M are independent for all parameter values, and π is globally identified.

Finally, if there exists no matching, then for any $r \times r$-submatrix M' of M, all terms in the Leibniz expansion are zero, and therefore $\det(M') = 0$ for any such submatrix. Then the rank of M is less than r, and its columns are linearly dependent for all values of the parameters, showing that π is structurally non-identified. $\square$

A.2 Proof of Theorem 2

Proof. As mentioned in the main text, the following adapts standard arguments in the theory of SNS matrices (see e.g. [8], Chapter 6). Note that the existence of a matching implies $n \geq r$. For any $R \subseteq X$ with $|R| = r$, let M_R denote the $r \times r$ submatrix of M obtained by retaining only the rows corresponding to elements of R. A matching $m : T \to R$ is identified with the permutation σ_m defined by $m(t_i) = x_{\sigma_m(i)}$. Condition (ii) of the theorem says that all matchings $m : T \to R$ have the same parity.

To prove sufficiency, assume there exists $R \subseteq X$ with $|R| = r$ such that: (i) there exists a matching $m : T \to R$; and (ii) every matching $m' : T \to R$ has the same parity as m. Consider any admissible parameter vector (π, M) (that is, M respects $\wedge$ and $\pi \in \Delta(T)$). We claim that M_R is nonsingular for every admissible choice of its positive entries consistent with $\wedge$. Indeed, expand $\det(M_R)$ by the Leibniz formula:

$$\det(M_R) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \prod_{l=1}^{r} M_R(\sigma(l), l).$$

A term is nonzero if and only if the corresponding assignment corresponds to a matching $T \to R$. By (i) at least one term is nonzero. By (ii), all nonzero terms correspond to matchings of the same parity, hence all have the same sign $\operatorname{sgn}(\sigma)$. Since every nonzero term is a product of positive entries, all nonzero terms add with the same sign, so the determinant cannot vanish:

$$\det(M_R) \neq 0.$$

Therefore M_R is invertible and π is globally identifiable.

For the necessity part, we prove the contrapositive. Assume no subset $R \subseteq X$ with $|R| = r$ satisfies conditions (i)–(ii). This means that, for every $R \subseteq X$ with $|R| = r$, either:

- (a) there is no matching $T \to R$; or
- (b) there exist at least two matchings $T \to R$ of opposite parity.

We will construct an admissible $n \times r$ matrix M^* (i.e. one respecting $\wedge$ and column-stochasticity) with $\operatorname{rank}(M^*) < r$. This will show that π is not globally identifiable. We proceed in two steps:

Step 1: for each R, build an admissible $n \times r$ matrix $M^{(R)}$ with $\det\big(M_R^{(R)}\big) = 0$.
Fix R with $|R| = r$.

Case (a): no matching $T \to R$. Then every permutation σ uses at least one zero entry of M_R, so every Leibniz monomial is zero and $\det(M_R) \equiv 0$ for all matrices with that zero pattern. Hence we can pick any admissible (column-stochastic) $M^{(R)}$ respecting $\wedge$ and obtain $\det(M_R^{(R)}) = 0$.

Case (b): two matchings of opposite parity exist. Let m_+ and m_- be two such matchings.

Fix a scalar $\lambda > 0$. Define an $r \times r$ matrix $\widetilde{M}_R(\lambda)$ respecting the same zero pattern as M_R by setting:

$$\widetilde{M}_R(\lambda)\big(\sigma_{m_+}(l), l\big) = \lambda \quad \text{for all } l,$$

all other allowed (non-forced-zero) entries in $\widetilde{M}_R(\lambda)$ equal to 1, and forced zeros equal to 0. Then, in $\det\big(\widetilde{M}_R(\lambda)\big)$, the monomial corresponding to m_+ equals $\operatorname{sgn}(\sigma_{m_+})\,\lambda^r$. Every other nonzero monomial contains at most $r-1$ factors equal to λ (because it differs from m_+ in at least one position). This implies that for λ large enough the sign of $\det\big(\widetilde{M}_R(\lambda)\big)$ coincides with $\operatorname{sgn}(\sigma_{m_+})$.

Analogously, define $\widehat{M}_R(\lambda)$ by placing λ along the matching m_- and 1 on all other allowed entries; then for λ sufficiently large, the sign of $\det\big(\widehat{M}_R(\lambda)\big)$ is $\operatorname{sgn}(\sigma_{m_-}) = -\operatorname{sgn}(\sigma_{m_+})$.

Next, we normalize columns to make them stochastic: let $D(\lambda)$ be the diagonal matrix whose ℓth diagonal entry is the sum of column ℓ of $\widetilde{M}_R(\lambda)$, and set

$$M_R^+(\lambda) = \widetilde{M}_R(\lambda)\, D(\lambda)^{-1}.$$

Similarly, define $M_R^-(\lambda)$ from $\widehat{M}_R(\lambda)$. It holds that

$$\det\big(M_R^+(\lambda)\big) = \det\big(\widetilde{M}_R(\lambda)\big) \cdot \det\big(D(\lambda)^{-1}\big),$$

and analogously for $\det\big(M_R^+(\lambda)\big)$. Since $\det(D(\lambda)^{-1}) > 0$, the sign is preserved, Thus, for λ large enough, $\det(M_R^+(\lambda)) > 0$ and $\det(M_R^-(\lambda)) < 0$.

Finally, we extend $M_R^+(\lambda)$ and $M_R^-(\lambda)$ to full $n \times r$ column-stochastic matrices M^+ and M^- respecting $\wedge$ by choosing arbitrary allowed positive entries on rows $X \setminus R$ and then renormalizing each column (again preserving the sign/nonzeroness of the minor on R up to a positive factor). In particular, we can ensure:

$$\det\big(M_R^+\big) > 0 \qquad \text{and} \qquad \det\big(M_R^-\big) < 0.$$

By the continuity of the determinant along convex combinations, there exists $\alpha \in (0,1)$ such that the matrix $M^{(R)}$ defined by

$$M^{(R)} := \alpha M^{+} + (1-\alpha)M^{-}$$

satisfies $\det\big(M_R^{(R)}\big) = 0$. Since the set of matrices that respect the zero pattern and column-stochasticity is clearly convex, $M^{(R)}$ is admissible.

This completes Step 1.

Step 2: by combining across the R, construct the desired admissible M^* with all $r \times r$ minors equal to 0.
Let $\mathcal{R} = \{R \subseteq X : |R| = r\}$. Define

$$M^* := \frac{1}{|\mathcal{R}|} \sum_{R \in \mathcal{R}} M^{(R)}.$$

Being a convex combination of admissible matrices, M^* is admissible.

Fix any $R_0 \in \mathcal{R}$. Consider the polynomial map that takes M to $\det(M_{R_0})$. This is multilinear in the rows indexed by R_0 and continuous. By construction, among the summands in M^*, the matrix $M^{(R_0)}$ has $\det\big(M_{R_0}^{(R_0)}\big) = 0$. Moreover, since in Step 1 we are free to choose each $M^{(R)}$ and we only require $\det(M_R^{(R)}) = 0$ for its own R, we may (and do) choose each $M^{(R)}$ so that all $r \times r$ minors vanish.[16] Hence every $r \times r$ minor of M^* is 0, implying

$$\operatorname{rank}(M^*) < r.$$

This means that the columns of M^* are linearly dependent and therefore π is not globally identified. □

A.3 Proof of Theorem 3

Proof. We will show that the columns of M are generically independent. Suppose that $a^* \in A$ separates types. Then there exists a matching $m : T \to X$ such that, for all $t \in T$, $m(t) = c_t(a^*)$. This implies that $n \geq r$. Form an $r \times r$-submatrix M' of M using the rows $\{k : x_k = m(t),\ t \in T\}$. The determinant of M' is a polynomial in the parameters $\{f(s)\}_{s \in A}$. Consider the point in the parameter space where $f(s) = 1$ if $s = a^*$ and $f(s) = 0$ otherwise. At this point, we have that M' is a permutation matrix, which implies $det(M') \neq 0$. Hence,

[16] One way to do this is: in Step 1, after obtaining an admissible $M^{(R)}$ with $\det(M_R^{(R)}) = 0$, replace it by a convex combination with a fixed admissible matrix of rank $< r$ respecting the same zero pattern (which exists under our standing assumption), so that all minors vanish while preserving admissibility.

the zero set of $det(M')$ is a proper subvariety and not the whole parameter space. We have just shown that there exists an $r \times r$-minor of M such that its zero set is a proper subvariety. This implies that the columns of M are generically independent. $\square$

A.4 Proof of Theorem 4

Proof. Fix $R \subseteq X$ with $|R| = r$. For $a \in A$, it holds that

$$M_R = \sum_{a \in A} f(a)\, M_R^a,$$

where M_R^a is the restriction of M^a to the rows in R, i.e. $M_R^a = (M^a)_R$. By the Leibniz formula, and identifying each permutation in S_r with the corresponding matching,

$$\det(M_R) = \sum_m \delta(m) \prod_{t \in T} M_R\big(m(t), t\big).$$

where the summation is over all matchings $m : T \to R$ and $\delta(m) \in \{-1, 1\}$ is determined by the parity of the permutation associated with m. For any matching $m : T \to R$ and any $t \in T$,

$$M_R\big(m(t), t\big) = \sum_{a \in A : c_t(a) = m(t)} f(a) = \sum_{\gamma : (m,\gamma) \in \mathcal{S}_R} f(\gamma(t)).$$

Expanding the product of sums and observing that each choice of admissible states $(\gamma(t))_{t \in T}$ uniquely defines a state-matching $(m, \gamma) \in \mathcal{S}_R$, we obtain

$$\det(M_R) = \sum_{(m,\gamma) \in \mathcal{S}_R} \delta(m) \prod_{t \in T} f\big(\gamma(t)\big), \tag{6}$$

i.e. the determinant is a polynomial in $\{f(a)\}_{a \in A}$ whose monomials are indexed precisely by state-matchings with image R.

Sufficiency. Assume there exists $R \subseteq X$, $|R| = r$, satisfying (i)–(ii). By (i), $\mathcal{S}_R \neq \varnothing$, hence (6) contains at least one monomial with nonzero coefficient. Condition (ii) rules out complete cancellation of all monomials: indeed, cancellation of (6) within every Γ-class would allow one to partition $\mathcal{S}_R$ into disjoint pairs $\big((m,\gamma),(m',\gamma')\big)$ with $\Gamma(m,\gamma) = \Gamma(m',\gamma')$ and $\delta(m') = -\delta(m)$ for every pair, contradicting (ii). Therefore, $\det(M_R)$ is not the zero polynomial. Hence M has generically full column rank r, and π is generically identifiable from $p = M\pi$.

Necessity. Suppose π is generically identifiable. Then M has generically full column rank

r, so there exists at least one $r \times r$ minor M_R whose determinant is not identically zero as a polynomial in $\{f(a)\}_{a \in A}$. Fix such an R. If $\mathcal{S}_R = \varnothing$, then every product in the Leibniz formula is identically zero, so $\det(M_R) \equiv 0$, a contradiction. Thus (i) holds.

If (ii) fails for this R, then there exists a partition of $\mathcal{S}_R$ into disjoint pairs $\big((m, \gamma), (m', \gamma')\big)$ such that $\Gamma(m, \gamma) = \Gamma(m', \gamma')$ and m' differs from m by an odd number of swaps, i.e. $\delta(m') = -\delta(m)$, for every pair. For each pair, $\Gamma(m, \gamma) = \Gamma(m', \gamma')$ implies

$$\prod_{t \in T} f\big(\gamma(t)\big) = \prod_{t \in T} f\big(\gamma'(t)\big),$$

while $\delta(m') = -\delta(m)$ implies the corresponding contributions in (6) cancel. Hence all terms cancel and $\det(M_R) \equiv 0$, again contradicting the choice of R. Therefore (ii) must hold for this R.

Structural non-identifiability. Suppose that no $R \subseteq X$ with $|R| = r$ satisfies (i)–(ii). Then, for every R, the polynomial $\det(M_R)$ is identically zero, hence every $r \times r$ minor of M vanishes for every parameter value. Thus, $\operatorname{rank}(M) < r$ for all admissible parameters, and π is structurally non-identifiable. $\square$

A.5 Proof of Theorem 5

Proof. Fix $R \subseteq X$ with $|R| = r$. From the determinant expansion established in the proof of Theorem 4, with M_R defined as there, we have

$$\det(M_R) = \sum_{(m,\gamma) \in \mathcal{S}_R} \delta(m) \prod_{t \in T} f\big(\gamma(t)\big), \tag{7}$$

where $\delta(m) \in \{-1, 1\}$ denotes the parity of the matching m.

Sufficiency. If condition (ii) holds, then $\delta(m)$ is constant over all $(m, \gamma) \in \mathcal{S}_R$. Hence every nonzero monomial in $\det(M_R)$ has the same sign. The condition $\mathcal{S}_R \neq \varnothing$ ensures that at least one such monomial exists, and therefore $\det(M_R) \neq 0$ for every state distribution f in the interior of the simplex. It follows that M_R is nonsingular for all admissible parameter values, and π is globally identifiable.

Necessity. Fix $R \subseteq X$ with $|R| = r$. If $\mathcal{S}_R = \varnothing$, then the determinant expansion contains no monomials and hence it is identically zero, in which case M_R is singular for every admissible state distribution. Therefore global identifiability requires $\mathcal{S}_R \neq \varnothing$.
Next, suppose condition (i) holds but condition (ii) fails. Then there exist two state-matchings $(m, \gamma), (m', \gamma') \in \mathcal{S}_R$ with opposite parity, i.e. $\delta(m) = -\delta(m')$, and in the expansion of

$\det(M_R)$ the monomials

$$\prod_{t\in T} f(\gamma(t)) \qquad \text{and} \qquad \prod_{t\in T} f(\gamma'(t))$$

appear with opposite signs.
Let

$$A^* := \{\gamma(t) : t \in T\} \cup \{\gamma'(t) : t \in T\}$$

be the set of states used by these two state-matchings. Fix a small $\varepsilon > 0$ and construct a strictly positive $f^+ \in \Delta(A)$ as follows: assign probability mass ε to each state in $A \setminus A^*$, assign probability mass ε to each state in $A^* \setminus \{\gamma(t) : t \in T\})$, and distribute the remaining probability mass uniformly over the set $\{\gamma(t) : t \in T\}$. In this way all components of f^+ are strictly positive. For ε small enough the contribution of the monomial indexed by (m, γ) dominates the sum of all remaining monomials in (7), so that we obtain $\det(M_R) > 0$ at the parameter values f^+.
Analogously, define a strictly positive $f^- \in \Delta(A)$ by making the states in $\{\gamma'(t) : t \in T\}$ receive the bulk of the probability mass and all other states probability of order ε. For ε small enough this yields $\det(M_R) < 0$ at the parameter values f^-.
Finally, observe that the set of strictly positive points in $\Delta(A)$ is convex, and by the continuity of the determinant map there exists a combination $f^* \in \Delta(A)$ of f^+ and f^- such that $\det(M_R) = 0$ at the parameter values f^*. We have shown that M_R is singular for some strictly positive f, so that π is not globally identifiable on the interior of the simplex. □

A.6 Proof of Corollary 1

Proof. Fix the separating state a^*. By the definition of a separating state, there exists a matching $m_{a^*} : T \to X$ with $m_{a^*}(t) = c_t(a^*)$. Write $R = \mathrm{Im}(m_{a^*})$. If there exists a possible reassignment φ that is odd, then for each $t \in T$ choose a state $\gamma(t) \in A$ such that

$$c_t(\gamma(t)) = c_{\varphi(t)}(a^*).$$

Define $m : T \to R$ by $m(t) := c_{\varphi(t)}(a^*)$. Since φ is bijective, $\mathrm{Im}(m) = R$, and (m, γ) is a state-matching for which m differs from m_a by an odd permutation. Hence, by Theorem 5, global identifiability fails.

Conversely, suppose global identifiability fails. We show that there exists a possible reassignment φ that is odd. By Theorem 5 there exist two state-matchings (m, γ) and (m_{a^*}, a^*) with $\mathrm{Im}(m) = R$ such that m differs from m_{a^*} by an odd number of swaps. Define $\varphi :=$

$m_{a^*}^{-1} \circ m$. Then φ is an odd permutation of T. Moreover, for each $t \in T$ we have

$$c_t(\gamma(t)) = m(t) = c_{\varphi(t)}(a^*),$$

so $t \rightsquigarrow \varphi(t)$ and φ is the required possible reassignment. $\square$

A.7 Proof of Theorem 6

Proof. $(i) \Rightarrow (ii)$: Suppose statement (ii) of the theorem fails, i.e., $\mathbf{x} \in \text{Null}(M^a) \cap \text{Null}(M^b)$ and $\mathbf{x} \neq 0$. Then,

$$M\mathbf{x} = (f(a)M^a + f(b)M^b)\mathbf{x} = f(a)M^a\mathbf{x} + f(b)M^b\mathbf{x} = \mathbf{0},$$

irrespective of the values of $f(a)$ and $f(b)$. Therefore, for all $f(a)$ and $f(b)$, the matrix $M = f(a)M^a + f(b)M^b$ is not invertible. Hence, M is not generically invertible, and the parameters in π are not generically identifiable.

$(ii) \Rightarrow (iii)$: Suppose statement (iii) of the theorem fails, i.e., both $c_{t_1}(a) = c_{t_2}(a)$ and $c_{t_1}(b) = c_{t_2}(b)$. This implies that the columns of M^a corresponding to t_1 and t_2 are the same vectors. Analogously, for M^b. Since these columns are one-hot vectors, it follows that

$$M^a, M^b \in \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

Note that irrespective of which of these two forms either of the two matrices M^a and M^b take, it holds that

$$\text{Null}(M^a) = \text{Null}(M^b) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

But, then $\text{Null}(M^a) \cap \text{Null}(M^b) \neq \{\mathbf{0}\}$. This concludes this part of the proof.

$(iii) \Rightarrow (i)$: Note that, WLOG we can assume that $c_{t_1}(a) \neq c_{t_2}(a)$. So, the columns of M^a corresponding to t_1 and t_2 are different vectors. Since these columns are one-hot vectors, this implies that

$$M^a \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

Observe that, any such M^a is invertible. As such, by Theorem 8, it follows that $M = f(a)M^a + f(b)M^b$ is generically invertible. This concludes the proof. $\square$

A.8 Proof of Theorem 7

Proof. $(i) \Rightarrow (ii)$: Suppose condition (ii) fails. First, suppose that $\text{Null}(M^a) \cap \text{Null}(M^b) \neq \{\mathbf{0}\}$. Then, fix any non-trivial $\mathbf{z}$, satisfying $\mathbf{z} \in \text{Null}(M^a)$ and $\mathbf{z} \in \text{Null}(M^b)$ and observe that

$$M\mathbf{z} = (f(a)M^a + f(b)M^b)\mathbf{z} = f(a)M^a\mathbf{z} + f(b)M^b\mathbf{z} = \mathbf{0}$$

Since $\mathbf{z}$ is non-trivial, this implies that $\text{Null}(M) \neq \{\mathbf{0}\}$. By the Invertible Matrix Theorem, it follows that M is not invertible.

Next, suppose that $\text{Null}(M^{a^T}) \cap \text{Null}(M^{b^T}) \neq \{\mathbf{0}\}$. Then, fix any non-trivial $\mathbf{w}$, satisfying $\mathbf{w} \in \text{Null}(M^{a^T})$ and $\mathbf{w} \in \text{Null}(M^{b^T})$ and observe that

$$M\mathbf{w} = (f(a)M^{a^T} + f(b)M^{b^T})\mathbf{w} = f(a)M^{a^T}\mathbf{w} + f(b)M^{b^T}\mathbf{w} = \mathbf{0}$$

Since $\mathbf{w}$ is non-trivial, this implies that $\text{Null}(M^T) \neq \{\mathbf{0}\}$. By the Invertible Matrix Theorem, it follows that M is not invertible. This concludes this part of the proof.

$(ii) \Rightarrow (iii)$: Suppose statement (iv) of the theorem fails, i.e., there exist $t, t' \in T$ such that, for all $a \in A$, it holds that $c_t(a) = c_{t'}(a)$, or, there exists an alternative $w \in X$ such that, for all $a \in A$ and all $t \in T$, it holds that $c_t(a) \neq w$.

First, consider the case of there existing $t, t' \in T$ such that, for all $a \in A$, it holds that $c_t(a) = c_{t'}(a)$. WLOG, we can assume that t and t' correspond to the first two columns of the matrices. Then, this implies that, for all $a \in A$, the first two columns of M^a are the same vectors. In other words, for all $a \in A$, it holds that

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in \text{Null}(M^a)$$

But then, $\text{Null}(M^a) \cap \text{Null}(M^b) \neq \{\mathbf{0}\}$.

Next, consider the case of there existing an alternative $w \in X$ such that, for all $a \in A$ and all $t \in T$, it holds that $c_t(a) \neq w$. WLOG, we can assume that this w corresponds to the last column of the matrices. Then, this implies that, for all $a \in A$, the last row of M^a is the zero vector. In other words, for all $a \in A$, it holds that

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathrm{Null}(M^{a^T})$$

But then, $\mathrm{Null}(M^{a^T}) \cap \mathrm{Null}(M^{b^T}) \neq \{\mathbf{0}\}$. This concludes this part of the proof.

$(iii) \Rightarrow (i)$: For all $t, t' \in T$ with $t \neq t'$, there exists an $a \in A$ such that $c_t(a) \neq c_{t'}(a)$, and, for all $w \in X$, there exist $a \in A$ and $t \in T$ such that $c_t(a) = w$. Note that this implies that not all terms in the Leibniz formula for the determinant of M are zero. Since the columns of M^a and M^b are one-hot vectors, this implies that at least one such term is unique in its form $f(a)^\alpha f(b)^{3-\alpha}$, for some $\alpha \in \{1, 2, 3\}$. But, then M is generically invertible. This concludes the proof. □

A.9 Proof of Theorem 8

Proof. Consider an $r \times r$ matrix $N = \sum_{b \in B} f(b) M^b$ which is generically invertible. By the Invertible Matrix Theorem, it follows that $\det(N) \neq 0$, for all $\{f(b)\}_{b \in B}$ but some Lebesgue measure zero set. Fix an arbitrary $r \times r$ matrix M^a and define $M(\alpha) = \alpha M^a + (1 - \alpha) N$ for all $\alpha \in [0, 1]$. Observe that $\det(M(\alpha))$ is a polynomial of degree at most r. Since $\det(M(0)) = \det(N) \neq 0$, for all $\{f(b)\}_{b \in B}$ but some Lebesgue measure zero set, it follows that $\det(M(\alpha))$ is not the zero polynomial for all such $\{f(b)\}_{b \in B}$. This implies that the polynomial vanishes for at most r values of α. Since r is finite, this implies that $\det(M(\alpha)) \neq 0$, for all $\{f(a)\}_{a \in A}$ but some Lebesgue measure zero set. By the Invertible Matrix Theorem, it follows that $M = \alpha M^a + (1 - \alpha) N$ is generically invertible. □

A.10 Proof of Corollary 2

Proof. Fix a probability measure $\overline{f}(a)$ on A such that $\overline{M} = \sum_{a \in A} \overline{f}(a) M^a$ is invertible. Now, observe that $M = \sum_{a \in A} f(a) M^a$, for any probability measure f on A, can be written as a convex combination of $\overline{M}$ and a matrix N which itself is some convex combination of the M^a's. In short, $M = \alpha \overline{M} + (1 - \alpha) N$, for some $\alpha \in (0, 1)$ and N given by $N = \sum_{a \in A} \underline{f}(a) M^a$, for some probability measure $\underline{f}$ on A. Since $\overline{M}$ is invertible, Theorem 8 implies that $M = \alpha \overline{M} + (1 - \alpha) N$ is generically invertible. Note that this is true for any such N. Hence, $M = \sum f(a) M^a$ is generically invertible. □

A.11 Proof of Theorem 9

Proof. $(i) \Rightarrow (ii)$: Let $A = \{a, b\}$ and suppose that $M = f(a)M^a + f(b)M^b$ is generically invertible. By the Invertible Matrix Theorem, it then follows that, for all $\{f(a)\}_{a \in A}$ but some Lebesgue-measure zero set, it holds that $\text{Null}(f(a)M^a + f(b)M^b) = \{\mathbf{0}\}$. Now, towards a proof by contradiction, suppose that there exists some $\mathbf{z}$ with $\mathbf{z} \neq \mathbf{x} + \mathbf{y}$ such that for some non-trivial $\mathbf{x} \in \text{Null}(M^a)$ and some non-trivial $\mathbf{y} \in \text{Null}(M^b)$, it holds that

$$(f(a)M^a + f(b)M^b)\mathbf{z} = M^a\mathbf{y} + M^b\mathbf{x}$$

Note that then there also exists some non-trivial $\mathbf{x} \in \text{Null}(M^a)$ and some non-trivial $\mathbf{y} \in \text{Null}(M^b)$ such that

$$(f(a)M^a + f(b)M^b)\mathbf{z} = f(a)M^a\mathbf{y} + f(b)M^b\mathbf{x}$$

Now, that this implies that

$$(f(a)M^a + f(b)M^b)\mathbf{w} = \mathbf{0}$$

for $\mathbf{w}$ defined as $\mathbf{w} = \mathbf{z} - \mathbf{x} - \mathbf{y}$. To see this, observe that

$$\begin{aligned}
& & \mathbf{0} &= (f(a)M^a + f(b)M^b)\mathbf{w} \\
\Leftrightarrow & & \mathbf{0} &= (f(a)M^a + f(b)M^b)(\mathbf{z} - \mathbf{x} - \mathbf{y}) \\
\Leftrightarrow & & (f(a)M^a + f(b)M^b)\mathbf{z} &= (f(a)M^a + f(b)M^b)(\mathbf{x} + \mathbf{y}) \\
\Leftrightarrow & & (f(a)M^a + f(b)M^b)\mathbf{z} &= f(a)M^a\mathbf{x} + f(a)M^a\mathbf{y} + f(b)M^b\mathbf{x} + f(b)M^b\mathbf{y} \\
\Leftrightarrow & & (f(a)M^a + f(b)M^b)\mathbf{z} &= f(a)M^a\mathbf{y} + f(b)M^b\mathbf{x}
\end{aligned}$$

where we used that $\mathbf{x} \in \text{Null}(M^a)$ implies that $f(a)M^a\mathbf{x} = \mathbf{0}$ and that $\mathbf{y} \in \text{Null}(M^b)$ implies that $f(b)M^b\mathbf{y} = \mathbf{0}$. Since $\mathbf{z} \neq \mathbf{x} + \mathbf{y}$, it follows that $\mathbf{w} = \mathbf{z} - \mathbf{x} - \mathbf{y} \neq \mathbf{0}$. As such, $(f(a)M^a + f(b)M^b)\mathbf{w} = \mathbf{0}$ implies that $\text{Null}(f(a)M^a + f(b)M^b) \neq \{\mathbf{0}\}$. By the Invertible Matrix Theorem, it follows that $M = (f(a)M^a + f(b)M^b) \in \mathcal{M}_{r \times r}(\mathbb{R})$ is not generically invertible. Hence, we have arrived at our desired contradiction. This concludes this part of the proof.

$(ii) \Rightarrow (iii)$: Suppose that condition (iii) in the statement of the theorem does not hold. This implies that there exists $t, t' \in T$ such that WLOG $c_t(a) = c_{t'}(a)$, while the split between $c_t(b)$ and $c_{t'}(b)$ is typical of $T \setminus \{t, t'\}$.

First, consider the case where $c_t(b) = c_{t'}(b)$. That is, Column t and t' of matrix M^b coincide. Since $c_t(a) = c_{t'}(a)$, also Column t and t' of matrix M^a coincide. It follows that there exists a non-trivial $\mathbf{x} \in \mathbb{R}^r$ such that $\mathbf{x} \in \text{Null}(M^a)$ and $\mathbf{x} \in \text{Null}(M^b)$. That is, $M^a\mathbf{x} = \mathbf{0}$ and $M^b\mathbf{y} = \mathbf{0}$. But then, fixing $\mathbf{z} = \mathbf{0}$ and $\mathbf{y} = \mathbf{x}$, we have some non-trivial a $\mathbf{x} \in \text{Null}(M^a)$, some non-trivial $\mathbf{y} \in \text{Null}(M^b)$, and some $\mathbf{z} \in \text{span}\{\mathbf{x}, \mathbf{y}\}^\perp$ such that

$$(f(a)M^a + f(b)M^b)\mathbf{z} = \mathbf{0} = f(a)M^a\mathbf{y} + f(b)M^b\mathbf{x}$$

Next, consider the case where $c_t(b) \neq c_{t'}(b)$. Definition 6 then implies that the set $T \setminus \{t, t'\}$ is non-empty. But, then the split between $c_t(b)$ and $c_{t'}(b)$ being typical of $T \setminus \{t, t'\}$ implies that

$$(\mathbf{e}^{c_t(b)} - \mathbf{e}^{c_{t'}(b)}) \in \text{span}\{M_{*j} : \text{column } j \text{ corresponds to some type } s \in T \setminus \{t, t'\}\}$$

where $\mathbf{e}^w$, for any $w \in \{x, y\}$, is defined such that, for all $i = 1, \ldots, r$

$$\mathbf{e}_i^w = \begin{cases} 1, & \text{if Row } i \text{ corresponds to alternative } w \\ 0, & \text{otherwise} \end{cases}$$

Defining $\mathbf{x} \in \mathbb{R}^r$ via

$$\mathbf{x}_i = \begin{cases} 1, & \text{if Row } i \text{ corresponds to alternative } c_t(b) \\ -1, & \text{if Row } i \text{ corresponds to alternative } c_{t'}(b) \\ 0, & \text{otherwise} \end{cases}$$

this implies that

$$M^b\mathbf{x} \in \text{span}\{M_{*j} : \text{column } j \text{ corresponds to some type } s \in T \setminus \{t, t'\}\}$$

In other words, there exists $\mathbf{w} \in \text{span}\{\mathbf{x}\}^\perp$ such that

$$(f(a)M^a + f(b)M^b)\mathbf{w} = f(b)M^b\mathbf{x} \tag{8}$$

By Theorem 8, there also exist $s, s' \in T$ such that $c_s(b) = c_{s'}(b)$, while the split between $c_s(a)$ and $c_{s'}(a)$ is typical of $T \setminus \{s, s'\}$. As such, analogously to above, we can define $\mathbf{y} \in \mathbb{R}^r$ via

$$\mathbf{y}_i = \begin{cases} 1, & \text{if Row } i \text{ corresponds to alternative } c_s(a) \\ -1, & \text{if Row } i \text{ corresponds to alternative } c_{s'}(a) \\ 0, & \text{otherwise} \end{cases}$$

such that there exists $\mathbf{v} \in \text{span}\{\mathbf{y}\}^{\perp}$ satisfying

$$(f(a)M^a + f(b)M^b)\mathbf{v} = f(a)M^a\mathbf{y} \tag{9}$$

Combing Equations (8) and (9), we find that there exists some non-trivial $\mathbf{x} \in \text{Null}(M^a)$, some non-trivial $\mathbf{y} \in \text{Null}(M^b)$, and some $\mathbf{z} \in \text{span}\{\mathbf{x}, \mathbf{y}\}^{\perp}$ such that

$$(f(a)M^a + f(b)M^b)\mathbf{z} = f(a)M^a\mathbf{y} + f(b)M^b\mathbf{x}$$

Note that $\mathbf{x} \in \text{Null}(M^a)$ implies that also $(f(b)\mathbf{x}) \in \text{Null}(M^a)$ and that $\mathbf{y} \in \text{Null}(M^b)$ implies that also $(f(a)\mathbf{y}) \in \text{Null}(M^b)$. Hence, this implies that there also exists some non-trivial $\mathbf{x} \in \text{Null}(M^a)$, some non-trivial $\mathbf{y} \in \text{Null}(M^b)$, and some $\mathbf{z} \in \text{span}\{\mathbf{x}, \mathbf{y}\}^{\perp}$ such that

$$(f(a)M^a + f(b)M^b)\mathbf{z} = M^a\mathbf{y} + M^b\mathbf{x}$$

$(iii) \Rightarrow (i)$: Fix some $a, b \in A$ with $a \neq b$ and suppose that, for all $t, t' \in T$ where $c_t(a) = c_{t'}(a)$, the split between $c_t(b)$ and $c_{t'}(b)$ is not typical of $(T \setminus \{t, t'\})$. For any such t and t', define $\mathbf{x} \in \mathbb{R}^r$ via

$$\mathbf{x}_i = \begin{cases} 1, & \text{if the } i\text{'th column of } M \text{ corresponds to type } t \\ -1, & \text{if the } i\text{'th column of } M \text{ corresponds to type } t' \\ 0, & \text{otherwise} \end{cases}$$

Then, $\mathbf{x} \in \text{Null}(M^a)$. Now, the split between $c_t(b)$ and $c_{t'}(b)$ not being typical of $(T \setminus \{t, t'\})$ implies that, for any such non-trivial $\mathbf{x}$ and all $\mathbf{z} \in \text{span}\{\mathbf{x}\}^{\perp}$, it holds that

$$M\mathbf{z} \neq M^b\mathbf{x}$$

Since $\mathbf{x} \in \text{Null}(M^a)$ and, thus, $M^a\mathbf{x} = \mathbf{0}$, this implies that, for $f(b) \neq 0$, it holds that

$$M\mathbf{z} \neq (f(a)M^a + f(b)M^b)\mathbf{x}$$

But, since any $\mathbf{w} \in \text{Null}(M)$ can be written as $\mathbf{w} = \mathbf{z} - \mathbf{x}$, for some $\mathbf{x} \in \text{Null}(M^a)$ and some $\mathbf{z} \in \text{span}\{\mathbf{x}\}^{\perp}$, this implies that the only such $\mathbf{w}$ is the trivial one, i.e., $\mathbf{w} = \mathbf{0}$ and thus $\text{Null}(M) = \{\mathbf{0}\}$. By the Invertible Matrix Theorem, it follows that M is invertible. Hence, the convex combination $f(a)M^a + f(b)M^b$ is generically invertible. $\square$

A.12 Proof of Theorem 10

Proof. As a first step, we will show that the Kruskal column rank[17] of M_j is at least v_j generically, for each $j \in \{1,2,3\}$. Take any v_j types in T and denote the set of these types by $\bar{T}$. From *(ii)*, it follows that there is an injective function $m : \bar{T} \to X_j$ such that $t \wedge_j m(t)$ for all $t \in \bar{T}$. Let $l(t)$ be such that $x_{l(t)} = m(t)$ and $k(t)$ be such that $t_{k(t)} = t$, for all $t \in \bar{T}$. Form a $v_j \times v_j$-submatrix of M_j using rows $\{l(t) : t \in \bar{T}\}$ and columns $\{k(t) : t \in \bar{T}\}$. Observe that the determinant of this matrix is a sum of monomials, one of which is $\prod_{t \in \bar{T}} M_j(l(t), k(t))$. The set of parameter values contains points where, for all $t \in T$, $M_j(l(t), k(t)) = 1$. At any of such points, the determinant of the submatrix is equal to 1. Hence, the zero set of this determinant is a proper subvariety and not the whole parameter space.

We have just shown that, for any selection of v_j columns of M_j, we can find a $v_j \times v_j$-minor of M_j formed of these columns such that its zero set is a proper subvariety. Hence, for any v_j columns of M_j, the columns are generically independent. Denote by Θ_j the union of the zero sets of these minors. Since this is a finite union, Θ_j is a proper subvariety too. For any point in the parameter space that lies outside of Θ_j, all combinations of v_j columns are independent. This implies that the Kruskal column rank of M_j is at least v_j generically.

For the second step, observe that outside of $\Theta_0 = \cup_j \Theta_j$ the Kruskal column rank of M_j is at least v_j for all $j \in \{1,2,3\}$. Denote by Θ the union of Θ_0 and the set of parameter values such that some component of π is equal to 0. Clearly, Θ is a proper subvariety. Outside Θ, the sum of the Kruskal column ranks of M_1, M_2, and M_3 is greater than or equal to $v_1 + v_2 + v_3$ and all components of π are strictly positive. Relying on the Kruskal's result [25], Corollary 2 of [1] says that, for $J = 3$, the parameters of the model (4) are uniquely identifiable, up to label swapping, if the sum of the Kruskal column ranks of M_1, M_2, and M_3 is greater than or equal to $2r + 2$. From this and assumption *(i)*, it follows that the parameters outside Θ are uniquely identifiable, up to label swapping. Hence, the parameters are generically identifiable, up to label swapping. □

A.13 Proof of Theorem 11

Proof. It follows from Theorem 4 that, for each j, there is a proper subvariety Θ_j of the parameter space such that outside Θ_j the Kruskal column rank of M_j is r. Hence, outside of $\Theta_0 = \cup_j \Theta_j$ the sum of the Kruskal column ranks of the matrices M_1, M_2, and M_3 is $3r$. Denote by Θ the union of Θ_0 and the set of parameter values such that some component of π

[17] The Kruskal column rank of M is the largest number d such that every set of d columns of M are independent.

is equal to 0. For any point outside Θ we can apply Corollary 2 of [1]. Since Θ is a proper subvariety, the result follows. $\square$

References

[1] Allman, Elizabeth S., Catherine Matias, and John A. Rhodes (2009) "Identifiability of Parameters in Latent Structure Models With Many Observed Variables," *Annals of Statistics* 37: 3099–3132.

[2] Apesteguia, José and Miguel-Anguel Ballester (2025) "A Measure of Behavioral Heterogeneity," *American Economic Journal: Microeconomics*, forthcoming.

[3] Apesteguia, José, Miguel-Anguel Ballester, and Jay Lu (2017) "Single-Crossing Random Utility Models," *Econometrica* 85: 661–674.

[4] Aumann, Robert J. (1962) "Utility Theory Without the Completeness Axiom," *Econometrica*: 445-462.

[5] Azrieli, Yaron and John Rehbeck (2022) "Marginal Stochastic Choice," *arXiv*: 2208.08492 [econ.TH].

[6] Barseghyan, Levon, Maura Coughlin, Francesca Molinari, and Joshua C. Teitelbaum (2021) Heterogeneous Choice Sets and Preferences, *Econometrica* 89(5): 2015-2048.

[7] Bassett, Lowell R., John S. Maybee, and James Quirk (1968) "Qualitative Economics and the Scope of the Correspondence Principle," *Econometrica* 36(3): 544–563.

[8] Brualdi, Richard A. and Bryan L. Shader (1995) "Matrices of Sign-Solvable Linear Systems," Cambridge University Press, Cambridge.

[9] Cappé, Olivier, Eric Moulines, and Tobias Rydén (2005) "Inference in Hidden Markov Models," Springer, New York.

[10] Caradonna, Peter P. and Christopher Turansick (2026) "Identification in Stochastic Choice," *arXiv*: 2408.06547 [econ.TH].

[11] Chambers, Christopher P. and Christopher Turansick (2025) "The Limits of Identification in Discrete Choice," *Games and Economic Behavior* 150: 537–551.

[12] Cox, David A., John Little, and Donal O'Shea (1997) "Ideals, Varieties, and Algorithms," 2nd ed, Springer, New York.

[13] Dardanoni, Valentino, Paola Manzini, Marco Mariotti, and Christopher J. Tyson (2020) "Inferring Cognitive Heterogeneity from Aggregate Choices," *Econometrica* 88(3): 1269-1296.

[14] Dardanoni, Valentino, Paola Manzini, Marco Mariotti, Henrik Petri, and Christopher J. Tyson (2024) "Mixture Choice Data: Revealing Preferences and Cognition," *Journal of Political Economy* 131(3): 687-715.

[15] Doignon, Jean-Paul and Kota Saito (2023) "Adjacencies on Random Ordering Polytopes and Flow Polytopes," *Journal of Mathematical Psychology*, 114: 102768.

[16] Dubra, Juan, Fabio Maccheroni, and Efe A. Ok. (2004) "Expected Utility Theory Without the Completeness Axiom," *Journal of Economic Theory* 115.1: 118-133.

[17] Ephraim, Yariv and Neri Merhav (2002) "Hidden Markov Processes," *IEEE Transactions on Information Theory* 48, 1518-1569.

[18] Filiz-Ozbay, Emel and Yusufcan Masatlioglu (2023) "Progressive Random Choice," *Journal of Political Economy* 131: 716–750.

[19] Fishburn, Peter C. (1998) "Stochastic Utility," in *Handbook of Utility Theory*, ed. by S. Barbera, P. Hammond, and C. Seidl, Kluwer Dordrecht; 273–318.

[20] Gorman, William M. (1968) "More Scope for Qualitative Economics," *The Review of Economic Studies* 31: 65-68.

[21] Hall, Philip (1935) "On Representatives of Subsets," *Journal of the London Mathematical Society* 1: 26–30.

[22] Koopmans, Tjalling C. and Olav Reisersøl (1950) "The identification of Structural Characteristics," *The Annals of Mathematical Statistics* 21: 165–181.

[23] Koopmans, Tjalling C. (1950) "Statistical Inference in Dynamic Economic Models," Wiley, New York.

[24] Kops, Christopher J., Paola Manzini, Marco Mariotti, and Illia Pasichnichenko (2025) "Revealing Features from Pareto Optimal Choice," Working paper.

[25] Kruskal, Joseph B. (1977) "Three-Way Arrays: Rank and Uniqueness of Trilinear Decompositions, With Application to Arithmetic Complexity and Statistics," *Linear Algebra and its Applications* 18: 95–138.

[26] Lancaster, Kelvin J. (1962) "The Scope of Qualitative Economics," *The Review of Economic Studies* 29(2): 99–123.

[27] Lancaster, Kelvin J. (1966) "A New Approach to Consumer Theory," *Journal of Political Economy* 74(2): 132-157.

[28] Lindsay, Bruce G. (1995) "Mixture Models: Theory Geometry and Applications," *NSF-CBMS Regional Conference Series in Probability and Statistics* 5, IMS, Hayward, CA.

[29] Lu, Jay (2019) "Bayesian Identification: A Theory for State-Dependent Utilities," *American Economic Review* 109(9): 3192-3228.

[30] Manzini, Paola and Marco Mariotti (2012) "Categorize Then Choose: Boundedly Rational Choice and Welfare," *Journal of the European Economic Association* 10.5: 1141-1165.

[31] Manzini, Paola and Marco Mariotti (2018) "Dual Random Utility Maximisation," *Journal of Economic Theory* 177: 162–182.

[32] Manzini, Paola, Marco Mariotti, and Henrik Petri (2019) "Corrigendum to "Dual Random Utility Maximisation" [J. Econ. Theory 177 (2018) 162–182]," *Journal of Economic Theory* 184: 104944.

[33] Masatlioglu, Yusufcan, Daisuke Nakajima, and Erkut Y. Ozbay (2012) "Revealed Attention," *American Economic Review* 102.5: 2183-2205.

[34] McLachlan, Geoffrey J. and David Peel (2000) "Finite Mixture Models," Wiley, New York.

[35] Rhodes, John A. (2010) "A Concise Proof of Kruskal's Theorem on Tensor Decomposition," *Linear Algebra and its Applications* 432: 1818-1824.

[36] Samuelson, Paul A. (1947) "Foundations of Economic Analysis," Harvard University Press, Cambridge, Massachusetts

[37] Turansick, Christopher (2022) "Identification in the Random Utility Model," *Journal of Economic Theory* 203,105489.

[38] Yildiz, Kemal (2023) "Foundations of Self-Progressive Choice Models," *arXiv*: 2212.13449 [econ.TH].