Structural Breaks and Permanent Trends

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January 2003

Discussion Paper No 03/545

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Abstract

For a multivariate time series model with structural breaks, explicit representations of the Beveridge-Nelson and Granger-Gonzalo-Proietti permanent trends are derived from the Johansen maximum likelihood estimates.

Keywords: Multivariate Time Series, Structural Breaks, Permanent Trends.

JEL Classification: C32, C51, E3

1 Introduction

A good deal of recent research in econometric time series modelling is concerned with the impact of structural breaks - breaks in mean and trends - in cointegrating systems (e.g., Johansen *et al* [10, 2000], Banerjee *et al* [2, 1998], Gregory and Hansen [9, 1996], Bai and Perron [3, 1998] and Bai and Perron [4, 2001]). Models of permanent stochastic trends which rely on the notion of cointegration and vector error correction are obviously affected by structural breaks and the object of this paper is to derive permanent trends in the presence of breaks. We consider a maximum of two breaks and obtain explicit expressions for the permanent trend definitions of Beveridge and Nelson [1, 1981], hereafter B-N, as extended in a multivariate context by King *et al* [11, 1991] and for the Granger and Gonzalo [8, 1995], hereafter G-G, definition as interpreted by Proietti [13, 1997]. The next section defines the model and derives the permanent trends for the case of structural breaks with the main algebraic results in an appendix.

2 The Multivariate Model with Structural Breaks

Suppose x_t is a $(p\times 1)$ vector of I(1) variables with cointegrating rank r. Suppose further that there are two breaks in the sample with T_1 observations in the first period, and $T_2 - T_1$ observations in the second period, and T observations

*I am indebted to Jonathan Temple for helpful discussions on material in this paper. The research was undertaken with the assistance of a University of Bristol Fellowship and an award from the ESRC. Any errors or omissions are my own responsibility.

altogether so that there are $T - T_2$ observations in the third period. Johansen *et al* [10, (2000)] derive a likelihood ratio test for cointegration in the presence of breaks in trend and mean at known points. In general the model can be written:

$$\Delta x_t = \theta_o \Xi_t + \sum_{j=1}^{k} \theta_j \Delta x_{t-j} + \beta(\alpha', \gamma') \begin{pmatrix} x_{t-1} \\ t\Xi_t \end{pmatrix} + \sum_{i=1}^{k+1} \sum_{j=2}^{3} \kappa_{ji} D_{jt-i} + \zeta_t \tag{1}$$

where $x_t = (x_{1t}, x_{2t}, ..., x_{pt})', D_{jt} = 1$ for $t = T_{j-1}$, with $T_o = 0$, and $D_{jt} = 0$ otherwise and $\Xi_t' = (\Xi_{1t}', \Xi_{2t}', \Xi_{3t}')$ with $\Xi_{jt} = 1$ for $T_{j-1} + k + 2 \leqq t \leqq T_j$ and zero otherwise and where $T_o = 0$. This specification allows for shifts in the intercepts of both the VECM and cointegrating equations (although they cannot be identified individually), in the term $\theta_o \Xi_t$ and shifts in the trend in the cointegrating equations only, in the term $\gamma' t \Xi_t$ with $\beta \gamma' = 0$. The Ξ_{jt}'s are dummies for the effective sample period for each sub-period and the D_{jt-i}'s have the effect of eliminating the first $k + 1$ residuals of each period from the likelihood, thereby producing the conditional likelihood function given the initial values in each period.

2.1 Permanent Stochastic Trends

2.1.1 Beveridge-Nelson

The definition of the B-N permanent trend component in a multivariate context is:

$$x_t^{BN-P} = x_t + \sum_{i=1}^{i=\infty} E_t(\Delta x_{t+i} - \mu_{\Delta x}) \tag{2}$$

see, for example, Cochrane [7, 1994]. To determine a solution for (2), write the VECM in (1) as

$$\Delta x_t = \mathcal{K}_o H_t + \sum_{j=1}^{k} \theta_j \Delta x_{t-j} + \beta v_{t-1} + \zeta_t. \tag{3}$$

where $\mathcal{K}_o = (\theta_o, \varkappa)$ where $\varkappa$ contains the κ_{ji} vectors, and:

$$H_t = \begin{bmatrix} \Xi_t \\ \mathcal{D}_t \end{bmatrix}$$

where $\mathcal{D}_t$ contains the $D_{jt-i}s$, and $v_{t-1} = \alpha' x_{t-1} + \gamma' t \Xi_t$. It follows that:

$$v_t = \alpha' x_t + \gamma'(t+1)\Xi_t = \alpha' \Delta x_t + \gamma' \Xi_t + v_{t-1}$$

and then:

$$v_t = \mathcal{K}_{oo} H_t + \alpha' \theta_1 \Delta x_{t-1} + \cdots + \alpha' \theta_k \Delta x_{t-k} + \left(I + \alpha' \beta\right) v_{t-1} + \alpha' \zeta_t \tag{4}$$

where:

$$\mathcal{K}_{oo} = \left(\alpha' \theta_o + \gamma', \alpha' \varkappa\right).$$

Defining $\theta(L) = I_p - \sum_{j=1}^{k} \theta_j L^j$, the model in (3) can also be written:

$$\theta(L)\Delta x_t = \mathcal{K}_o H_t + \beta v_{t-1} + \zeta_t. \tag{5}$$

Appending (4) to the system in (3) we have a first order stationary vector autoregression of the form:

$$z_t = A_o H_t + A_1 z_{t-1} + \Psi \zeta_t \quad t = 1, \ldots, T \tag{6}$$

where z_t' is the $(1 \times pk + r)$ vector:

$$z_t' = (\Delta x_t', \Delta x_{t-1}', ..., \Delta x_{t-k+1}', v_t').$$

The matrices A_o and A_1 are defined as:

$$A_o = \begin{bmatrix} \mathcal{K}_o \\ 0 \\ 0 \\ \vdots \\ 0 \\ \mathcal{K}_{oo} \end{bmatrix}$$

and:

$$A_1 = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{k-1} & \theta_k & \beta \\ I & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 \\ \alpha'\theta_1 & \alpha'\theta_2 & \cdots & \alpha'\theta_{k-1} & \alpha'\theta_k & \alpha'\beta + I \end{bmatrix} \tag{7}$$

and:

$$\Psi = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \\ \alpha' \end{bmatrix}.$$

From (6) it follows that:

$$E(z_t) = \mu_z = (I - A_1)^{-1} A_o H_t \tag{8}$$

so that:

$$z_t - \mu_z = (I - A_1 L)^{-1} \Psi \zeta_t. \tag{9}$$

Define the matrix:

$$G = \begin{bmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then $G'z_t$ selects out Δx_t and it follows from (6) that:

$$\Delta x_t - \mu_{\Delta x} = G'(z_t - \mu_z) = G'\left(I - A_1 L\right)^{-1}\Psi\zeta_t = C(L)\zeta_t \tag{10}$$

which is the moving average representation. Inverting $[I - A_1]$, see Appendix, it is straightforward to show that[1]:

$$C(1) = G'[I - A_1]^{-1}\Psi = \theta(1)^{-1} - \theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1}\alpha'\theta(1)^{-1}$$

where $\theta(1) = I_p - \sum_1^k \theta_i$.

The expectations term in equation (2), can then be written:

$$\sum_{i=1}^{i=\infty} E_t(\Delta x_{t+i} - \mu_{\Delta x}) = G'A_1[I - A_1]^{-1}\left(z_t - \mu_z\right). \tag{11}$$

Some algebra, see Appendix, produces:

$$G'A_1\left(I - A_1\right)^{-1} = \left[C(1)\sum_1 \theta_i, C(1)\sum_2 \theta_i, C(1)\sum_3 \theta_i, \cdots C(1)\theta_k, -Q\right]. \tag{12}$$

where $Q = \theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1}$.

Let:

$$\theta^*(L) = \sum_1 \theta_i + \sum_2 \theta_i L + ... + \theta_k L^{k-1}$$

then from (11) obtain:

$$x_t^{BN-P} = x_t + C(1)\theta^*(L)\left(\Delta x_t - \mu_{\Delta x}\right) - Q\left(v_t - \mu_v\right). \tag{13}$$

Substituting $\theta^*(L)(1-L) = \theta(L) - \theta(1)$ and $v_t = \alpha' x_t + \gamma'(t+1)\Xi_t$ into (13) gives:

$$\begin{aligned} x_t^{BN-P} &= x_t - Q\alpha' x_t - Q\gamma'(t+1)\Xi_t + C(1)\left[\theta(L) - \theta(1)\right] x_t - C(1)\theta^*(1)\mu_{\Delta x} + Q\mu_v \\ &= C(1)\theta(1)x_t - Q\gamma'(t+1)\Xi_t + C(1)\theta(L)x_t - C(1)\theta(1)x_t - C(1)\theta^*(1)\mu_{\Delta x} + Q\mu_v \end{aligned}$$

$$\therefore\ x_t^{BN-P} = C(1)\theta(L)x_t - Q\gamma'(t+1)\Xi_t + \delta_o \tag{14}$$

with $\delta_o = -C(1)\theta^*(1)\mu_{\Delta x} + Q\mu_v$.

$\mu_{\Delta x}$ and μ_v are the means of stationary variables and can be estimated from sample counterparts. Explicit formulae can be deduced from (8), i.e.:

$$G'\mu_z = \mu_{\Delta x} = G'\left(I - A_1\right)^{-1} A_o H_t$$

[1] Proietti [13, 1997] obtains the same result using the Kalman filter except that instead of $\Theta(1)^{-1}$ he has $\left(\Theta(1) - \beta\alpha'\right)^{-1}$ but it is easy to show that the two forms give exactly the same $C(1)$.

and:

$$J'\mu_z = \mu_v = J'\left(I - A_1\right)^{-1} A_o H_t$$

where $J' = (0, 0, ..., 0, I)$ selects out the rows of μ_z associated with μ_v. Multiplying out the matrices using the results in the appendix yields:

$$\begin{aligned}\mu_{\Delta x} &= G'\left(I - A_1\right)^{-1} A_o H_t \\ &= \left(C(1)\theta_o - \theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1}\gamma'\right)\Xi_t + C(1)\varkappa\mathcal{D}_t \\ &= C(1)\mathcal{K}_o H_t - Q\gamma'\Xi_t\end{aligned}$$

and:

$$\mu_v = J'\left(I - A_1\right)^{-1} A_o H_t = -(\alpha'\theta(1)^{-1}\beta)^{-1}\left[(\alpha'\theta(1)^{-1}\theta_o + \gamma')\Xi_t + \alpha'\theta(1)^{-1}\varkappa\mathcal{D}_t\right]$$

As a first difference:

$$\Delta x_t^{BN-P} = C(1)\theta(L)\Delta x_t - Q\gamma'\Xi_t \tag{15}$$

and substituting the VECM in (5) into (15) we obtain:

$$\Delta x_t^{BN-P} = C(1)\left(\mathcal{K}_o H_t + \beta v_{t-1} + \zeta_t\right) - Q\gamma'\Xi_t$$

but, since $C(1)\beta = 0$, it follows that:

$$\begin{aligned}\Delta x_t^{BN-P} &= C(1)\mathcal{K}_o H_t + C(1)\zeta_t - Q\gamma'\Xi_t \\ &= \mu_{\Delta x} + C(1)\zeta_t\end{aligned} \tag{16}$$

so that the trends have no long run impact on the permanent component. The equivalent definitions of trend growth in (15) and (16) derived from the B-N decomposition are used by King *et al* [11, 1991] and Cochrane [7, 1994] for the case of no structural breaks.

The definition of trend from the B-N decomposition can be justified on the grounds that it is only permanent shocks which impact on the trend. That is, suppose, without any loss of generality that the non-singular matrix Γ_o decomposes the error term ζ_t into permanent and transitory shocks, i.e.:

$$\Gamma_o\zeta_t = \varepsilon_t = \begin{pmatrix}\varepsilon_t^P \\ \varepsilon_t^T\end{pmatrix}. \tag{17}$$

We can then decompose (16) into the model:

$$\begin{aligned}\Delta x_t^{BN-P} &= \mu_{\Delta x} + C(1)\Gamma_o^{-1}\Gamma_o\zeta_t \\ &= \mu_{\Delta x} + \Gamma(1)\varepsilon_t\end{aligned}$$

and since it is only permanent shocks that have an impact on the trend, it must be the case that:

$$C(1)\Gamma_o^{-1} = \Gamma(1) = [\Gamma_1, 0] \tag{18}$$

where Γ_1 is $(p \times p - r)$ and 0 is $(p \times r)$, where p is the number of variables in the system and r is the dimension of the cointegrating space. It follows that:

$$\Delta x_t^{BN-P} = \mu_{\Delta x} + C(1)\zeta_t = \mu_{\Delta x} + \Gamma_1\varepsilon_t^P. \tag{19}$$

and so it is only permanent shocks which have a long run impact on the trend.

2.1.2 Gonzalo and Granger

The B-N definition of trend has been criticised by Quah [6, 1989] and Lippi and Reichlin [12, 1994] because it does not contain any changes in permanent and transitory shocks. Gonzalo and Granger, hereafter G-G, [8, 1995] suggest a method of decomposing the series into permanent and transitory components to incorporate some shock dynamics into the permanent component which, in essence, allows CHANGES in the transitory component to have an impact on CHANGES in the permanent component and thus a transitory impact on the LEVEL of the permanent component. Their procedure is to decompose x_t into a permanent component, x_t^{GG-P}, which is a linear function of the common factors in the model, f_t, and a stationary transitory component, x_t^{GG-T}, i.e.:

$$x_t = x_t^{GG-P} + x_t^{GG-T} = \mathcal{A}_1 f_t + \widetilde{x}_t.$$

In a model without intercepts or trends they define the common factors as:

$$f_t = \beta'_\perp x_t$$

where $\beta_\perp$ represents the orthogonal complement of β, such that $\beta'_\perp \beta = 0$, and the transitory component as a linear function of the stationary error correction terms:

$$x_t^{GG-T} = \widetilde{x}_t = \mathcal{A}_2 \alpha' x_t$$

which ensures that the only linear combinations of x_t on which $\widetilde{x}_t$ has no long run impact will be $\beta'_\perp x_t$. To see this multiply through equation (1) by $\beta'_\perp$.

Since:

$$x_t = \mathcal{A}_1 \beta'_\perp x_t + \mathcal{A}_2 \alpha' x_t$$

inverting the matrix:

$$\begin{bmatrix} \beta'_\perp \\ \alpha' \end{bmatrix}$$

yields:

$$(\mathcal{A}_1, \mathcal{A}_2) = \left(\alpha_\perp (\beta'_\perp \alpha_\perp)^{-1}, \beta(\alpha'\beta)^{-1} \right).$$

where $\alpha_\perp$ is the orthogonal complement of α.

Adding a broken trend in the cointegrating vector yields a transitory component of the form:

$$x_t^{GG-T} = \beta(\alpha'\beta)^{-1}\alpha' x_t + \beta(\alpha'\beta)^{-1}\gamma'(t+1)\Xi_t$$

so the permanent component becomes:

$$x_t^{GG-P} = \alpha_\perp(\beta'_\perp \alpha_\perp)^{-1}\beta'_\perp x_t - \beta(\alpha'\beta)^{-1}\gamma'(t+1)\Xi_t$$

so that, in first differences:

$$\Delta x^{GG-P} = \alpha_\perp(\beta'_\perp \alpha_\perp)^{-1}\beta'_\perp \Delta x_t - \beta(\alpha'\beta)^{-1}\gamma'\Xi_t. \tag{20}$$

Substitute the moving average representation in (10), i.e.:

$$\Delta x_t = \mu_{\Delta x} + C(L)\zeta_t$$

into equation (20) to obtain:

$$\Delta x^{GG-P} = \alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp\mu_{\Delta x} + \alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp C(L)\zeta_t - \beta(\alpha'\beta)^{-1}\gamma'\Xi_t. \quad (21)$$

Then, since:

$$\alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp\mu_{\Delta x} = \alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp \left(C(1)\mathcal{K}_o H_t - Q\gamma'\Xi_t\right)$$

and $\alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp C(1)$=$C(1)$ and:

$$\begin{aligned}\alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp Q &= \left(I - \beta(\alpha'\beta)^{-1}\alpha'\right)Q \\ &= Q - \beta(\alpha'\beta)^{-1}\end{aligned}$$

it follows that:

$$\begin{aligned}\alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp\mu_{\Delta x} &= C(1)\mathcal{K}_o H_t - Q\gamma'\Xi_t + \beta(\alpha'\beta)^{-1}\gamma'\Xi_t \\ &= \mu_{\Delta x} + \beta(\alpha'\beta)^{-1}\gamma'\Xi_t.\end{aligned}$$

Substituting this result into (21) yields:

$$\Delta x^{GG-P} = \mu_{\Delta x} + \alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp C(L)\zeta_t.$$

Then using:

$$C(L) = C(1) + (1-L)C^*(L)$$

we obtain:

$$\begin{aligned}\Delta x^{GG-P} &= \mu_{\Delta x} + \alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp \left(C(1) + (1-L)C^*(L)\right)\zeta_t \\ &= \mu_{\Delta x} + C(1)\zeta_t + \alpha_\perp(\beta'_\perp\alpha_\perp)^{-1}\beta'_\perp C^*(L)(1-L)\Gamma_o^{-1}\varepsilon_t \\ &= \mu_{\Delta x} + \Gamma_1\varepsilon_t^P + \omega_1(L)\Delta\varepsilon_t^P + \omega_2(L)\Delta\varepsilon_t^T. \quad (22)\end{aligned}$$

using the results in (17) and (18).

Comparing (22) with (19) we see that $\Delta x^{GG-P} = \Delta x^{BN-P} + \omega_1(L)\Delta\varepsilon_t^P + \omega_2(L)\Delta\varepsilon_t^T$ so that short run dynamics in both permanent and transitory components have a short run impact in the Granger-Gonzalo formulation.

Proietti [13, 1997] noticed that the B-N decomposition can be amended to form a G-G decomposition by replacing the term $\theta(L)$ in equation (14) with:

$$\theta(L) = \theta(1) + (1-L)\theta^*(L)$$

to give:

$$x_t^{BN-P} = C(1)\theta(1)x_t + C(1)\theta^*(L)\Delta x_t - Q\gamma'(t+1)\Xi_t + \delta_o. \quad (23)$$

Adding the term -$C(1)\theta^*(L)\Delta x_t$ to (23) results in:

$$x_t^{GGP-P} = C(1)\theta(1)x_t - Q\gamma'(t+1)\Xi_t + \delta_o \tag{24}$$

which includes error dynamics. To see this, take first differences:

$$\Delta x_t^{GGP-P} = C(1)\theta(1)\Delta x_t - Q\gamma'\Xi_t$$

and substitute the moving average representation to give:

$$\begin{aligned}
\Delta x_t^{GGP-P} &= C(1)\theta(1)\mu_{\Delta x} + C(1)\theta(1)C(L)\zeta_t - Q\gamma'\Xi_t \\
&= C(1)\theta(1)\left[C(1)\mathcal{K}_o H_t - Q\gamma'\Xi_t\right] + C(1)\theta(1)C(L)\zeta_t - Q\gamma'\Xi_t \\
&= C(1)\mathcal{K}_o H_t - Q\gamma'\Xi_t + C(1)\theta(1)C(L)\zeta_t \\
&= \mu_{\Delta x} + C(1)\theta(1)C(L)\zeta_t
\end{aligned}$$

since $C(1)\theta(1)C(1) = C(1)$ and $C(1)\theta(1)Q = 0$. Then using $C(L) = C(1) + (1-L)C^*(L)$ we obtain:

$$\begin{aligned}
\Delta x_t^{GGP-P} &= \mu_{\Delta x} + C(1)\theta(1)\left(C(1) + (1-L)C^*(L)\right)\zeta_t \\
&= \mu_{\Delta x} + C(1)\zeta_t + \theta(1)(1-L)C^*(L)\Gamma_o^{-1}\varepsilon_t \\
&= \mu_{\Delta x} + \Gamma_1\varepsilon_t^P + \omega_1^*(L)\Delta\varepsilon_t^P + \omega_2^*(L)\Delta\varepsilon_t^T \qquad (25)
\end{aligned}$$

where we have used the same decomposition of the equation error ζ_t into transitory and permanent shocks as in (17). The expression in (25) contains changes in both permanent and transitory shocks and is only different from (22) by the weights on these short run dynamic components.

Appendix

Theorem 1 *If*

$$A_1 = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{k-1} & \theta_k & \beta \\ I & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 \\ \alpha'\theta_1 & \alpha'\theta_2 & \cdots & \alpha'\theta_{k-1} & \alpha'\theta_k & \alpha'\beta + I \end{bmatrix}$$

and:

$$G = \begin{bmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

then:

$$G'(I - A_1)^{-1} = \theta(1)^{-1} - \theta(1)^{-1} H \left(\sum_1 \theta_i \theta(1)^{-1} \right), \left(\theta(1)^{-1} - \theta(1)^{-1} H \theta(1)^{-1}\right) \sum_2 \theta_i, \cdots$$
$$\cdots, \left(\theta(1)^{-1} - \theta(1)^{-1} H \theta(1)^{-1}\right) \theta_k, -\theta(1)^{-1} \beta (\alpha' \theta(1)^{-1} \beta)^{-1}$$

where $H = \beta(\alpha'\theta(1)^{-1}\beta)^{-1}\alpha'$ *and* $\theta(1) = I_p - \sum_1^k \theta_i$.

Proof

Let

$$\Theta = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_{k-1} & \theta_k \\ I_p & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_p & 0 \end{bmatrix}$$

$$\beta_k' = \left[\beta' \, 0 \cdots 0\right]$$
$$\alpha_k' = \left[\alpha' \, 0 \cdots 0\right]$$

Then:

$$A_1 = \begin{bmatrix} \Theta & \beta_k \\ \alpha_k'\Theta & I_r + \alpha_k'\beta_k \end{bmatrix}$$

and so:

$$I_{(pk+r)} - A_1 = \begin{bmatrix} I_{pk} - \Theta & -\beta_k \\ -\alpha_k'\Theta & -\alpha_k'\beta_k \end{bmatrix}.$$

By direct inversion:

$$(I - A_1)^{-1} = \begin{bmatrix} B^{-1} - B^{-1}\beta_k(\alpha_k' B^{-1}\beta_k)^{-1}\alpha_k'\Theta B^{-1} & -B^{-1}\beta_k(\alpha_k' B^{-1}\beta_k)^{-1} \\ -(\alpha_k' B^{-1}\beta_k)^{-1}\alpha_k'\Theta B^{-1} & -(\alpha_k' B^{-1}\beta_k)^{-1} \end{bmatrix}$$

where $B = I_{pk} - \Theta$. We require the first p rows of $(I - A_1)^{-1}$, i.e. $G'(I - A_1)^{-1}$. By direct inversion we obtain:

$$B^{-1} = (I_{pk} - \Theta)^{-1} = E + F$$

where:

$$E = \begin{bmatrix} \theta(1)^{-1} & \theta(1)^{-1}\sum_2^k \theta_i & \theta(1)^{-1}\sum_3^k \theta_i & \cdots & \theta(1)^{-1}\theta_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta(1)^{-1} & \theta(1)^{-1}\sum_2^k \theta_i & \theta(1)^{-1}\sum_3^k \theta_i & \cdots & \theta(1)^{-1}\theta_k \end{bmatrix}$$

where:

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & I_p & 0 & 0 & \cdots & 0 \\ 0 & I_p & I_p & 0 & \cdots & 0 \\ 0 & I_p & I_p & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & I_p & I_p & I_p & I_p & I_p \end{bmatrix}$$

It follows that:

$$G'(I - A_1)^{-1} = G'(E+F) - G'(E+F)\beta_k(\alpha_k'(E+F)\beta_k)^{-1}\alpha_k'\Theta(E+F), \\ -G'(E+F)\beta_k(\alpha_k'(E+F)\beta_k)^{-1} \tag{A1}$$

or

$$G'(I - A_1)^{-1} = \theta(1)^{-1} - \theta(1)^{-1}H\left(\sum_1 \theta_i\theta(1)^{-1}\right), \left(\theta(1)^{-1} - \theta(1)^{-1}H\theta(1)^{-1}\right)\sum_2 \theta_i, \cdots \\ \cdots, \left(\theta(1)^{-1} - \theta(1)^{-1}H\theta(1)^{-1}\right)\theta_k, -\theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1} \tag{A2}$$

where $H = \beta(\alpha'\theta(1)^{-1}\beta)^{-1}\alpha'$.

Corollary 1. It follows that:

$$\begin{aligned} C(1) &= G'(I - A_1)^{-1}\Psi \\ &= \theta(1)^{-1} - \theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1}\alpha'\left(\sum_1 \theta_i\theta(1)^{-1}\right) - \theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1}\alpha' \end{aligned}$$

and so:

$$C(1) = G'(I - A_1)^{-1}\Psi = \theta(1)^{-1} - \theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1}\alpha'\theta(1)^{-1}$$

Corollary 2. Noting that $G'A_1(I - A_1)^{-1} = G'(I - A_1)^{-1} - G'$ from (A2) we obtain:

$$G'A_1(I - A_1)^{-1} = C(1)\sum_1 \theta_i, C(1)\sum_2 \theta_i, \cdots C(1)\theta_k, -\theta(1)^{-1}\beta(\alpha'\theta(1)^{-1}\beta)^{-1} \tag{A3}$$

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