Higher Order Asymptotic Theory for Semiparametric Estimation of Spectral Parameters of Stationary Processes

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Abstract

Let $g(\lambda)$ be the spectral density of a stationary process, and let $f_\theta(\lambda)$, $\theta \in \Theta$, be a fitted spectral model for $g(\lambda)$. A semiparametric estimator $\hat{\theta}_n$ of θ is given by minimizing a distance $D(f_\theta; \hat{g}_n)$ between f_θ and $\hat{g}_n$, where $\hat{g}_n$ is a nonparametric spectral density estimator based on n observations. Then it is known that $\hat{\theta}_n$ is asymptotically Gaussian efficient if $g(\lambda) = f_\theta(\lambda)$. Since there are infinitely many candidates for $D(f_\theta; \hat{g}_n)$, this paper discusses higher order asymptotic theory for $\hat{\theta}_n$ in relation to the choice of D. First, the second-order Edgeworth expansion for $\hat{\theta}_n$ is derived. Then it is shown that the bias-adjusted version of $\hat{\theta}_n$ is not second-order asymptotically efficient in general. Hence, for semiparametric estimation, the statement "first-order efficiency implies second-order efficiency" does not hold in general although it does holds true for the usual parametric estimation.

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1 Introduction

It is known that many semiparametric estimators based on n observations have been shown to be root-n-consistent and asymptotically normal. In the context of time series analysis where the process concerned has the spectral density $g(\lambda)$, Taniguchi (1987) considered fitting a family of parametric spectral densities $f_\theta(\lambda)$, $\theta \in \Theta$, and proposed an estimator $\hat{\theta}_n$ of θ by minimizing $\int_{-\pi}^{\pi} K\{f_\theta(\lambda)/\hat{g}_n(\lambda)\}d\lambda$ with respect to θ, where $K(x)$ is a smooth function having a unique minimum at $x = 1$, and $\hat{g}_n(\lambda)$ is a nonparametric estimator of $g(\lambda)$. He then showed that $\sqrt{n}(\hat{\theta}_n - \theta)$ is asymptotically normal and asymptotically Gaussian efficient if $g(\lambda) = f_\theta(\lambda)$. Since there are infinitely many candidates for $K(x)$, we can construct infinitely many first-order asymptotically efficient estimators $\hat{\theta}_n$. This motivates the present investigation and distinguish between these estimators in terms of their higher order asymptotic properties and efficiency.

For a class of nonparametric spectral density estimators $\hat{g}_n(\lambda)$, Bentkus and Rudzkis (1982) gave the second-order Edgeworth expansion for $\hat{g}_n(\lambda)$. Also Velasco and Robinson (1998) derived the Edgeworth expansion of the means of a Gaussian stationary process, which was studentized by $\sqrt{2\pi\hat{g}_n(0)}$.

In the context of semiparametric estimation, it was Robinson (1995) who first provided a higher order asymptotic result. For semiparametric averaged derivative estimators based on independent observations, he established a version of the Berry–Esseen theorem, which corresponds to the second-order asymptotic theory.

In this paper we will develop higher order asymptotic theory for the estimator $\hat{\theta}_n$ proposed by Taniguchi (1987). First, we give the second-order Edgeworth expansion of $\hat{\theta}_n$, and show that the bias-adjusted version $\hat{\theta}_n^*$ of $\hat{\theta}_n$ is not second-order asymptotically efficient in general. This depends on the choice of distance D, and we provide verifiable conditions on D such that second order efficiency is implied. In the usual parametric estimation theory it is known that "first-order efficiency implies second-order efficiency" (e.g., Ghosh (1994, p. 57)). This paper therefore establishes a sharp and interesting contrast between the usual parametric estimation and semiparametric estimation.

2 Minimum Contrast Estimation

Let $\{X_t\}$ be a Gaussian stationary process with mean zero, autocovariance function $R(\cdot)$, and spectral density $g(\lambda)$, $\lambda \in [-\pi, \pi]$. We fit some parametric family $\mathcal{P} = \{f_\theta : \theta = (\theta^1, \ldots, \theta^p)' \in \Theta \subset R^p\}$ of spectral densities to the true spectral density $g(\lambda)$ by minimizing a criterion. Initially, we make the following assumption.

Assumption 1. (i) The spectral density $g(\lambda)$ belongs to the class of spectral densities defined by

$\mathcal{F} \equiv \{g : g(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} G(j)e^{-ij\lambda}\right|^2$, there exist $C < \infty$ and $\delta > 0$ such that $\sum_{j=0}^{\infty}(1 + j^2)|G(j)| \le C$, $\left|\sum_{j=0}^{\infty} G(j)z^j\right| \ge \delta$ for all $|z| \le 1\}$.

(ii) The fitted model $f_\theta(\lambda)$ belongs to $\mathcal{F}$, and is four times continuously differentiable with respect to θ.

(iii) $K(x)$ is a four times continuously differentiable function on $(0, \infty)$, and has a unique minimum at $x = 1$.

Taniguchi (1987) introduced the criterion

$$D(f_\theta, g) \equiv \int_{-\pi}^{\pi} K\left\{\frac{f_\theta(\lambda)}{g(\lambda)}\right\} d\lambda, \tag{1}$$

which measures the nearness to f_θ to g. The following (E1)–(E3) are typical examples.

$$K(x) = \log x + \frac{1}{x}, \quad D(f_\theta, g) = \int_{-\pi}^{\pi}\left\{\log\frac{f_\theta(\lambda)}{g(\lambda)} + \frac{g(\lambda)}{f_\theta(\lambda)}\right\} d\lambda. \tag{E1}$$

This criterion is equivalent to the quasi-Gaussian maximum likelihood type criterion $\int_{-\pi}^{\pi}\{\log f_\theta(\lambda) + g(\lambda)/f_\theta(\lambda)\}d\lambda$ (Hosoya and Taniguchi (1982), Taniguchi and Kakizawa (2000)).

$$K(x) = (\log x)^2, \quad D(f_\theta, g) = \int_{-\pi}^{\pi}\{\log f_\theta(\lambda) - \log g(\lambda)\}^2 d\lambda. \tag{E2}$$

This is given in Taniguchi (1979) and (1981).

$$K(x) = (x^\alpha - 1)^2, \; 0 < \alpha < \infty, \quad D(f_\theta, g) = \int_{-\pi}^{\pi}\left\{\frac{f_\theta(\lambda)^\alpha}{g(\lambda)^\alpha} - 1\right\}^2 d\lambda. \tag{E3}$$

This defines a family of criterion functions parameterized by α.

Let us define a pseudo-true value of θ by

$$\theta(g) = \arg\min_{\theta \in \Theta} D(f_\theta; g). \tag{2}$$

In order to estimate $\theta(g)$, we need to estimate $g(\lambda)$ since $g(\lambda)$ is unknown. For this purpose we will use a nonparametric window type estimator, whose window functions $W(\cdot)$, $W_n(\cdot)$, and $w(\cdot)$ satisfy the following assumptions.

Assumption 2. (i) $W(x)$ is bounded, even, nonnegative and such that

$$\int_{-\infty}^{\infty} W(x)dx = 1; \quad \text{and} \quad \int_{-\infty}^{\infty} x^4 W(x)dx < \infty.$$

(ii) $w(x)$ is a continuous, even function with $w(0) = 1$ and $w(\pm 1) = 0$, and satisfies

$$\begin{cases} |w(x)| \le 1; \\ \int_{-\infty}^{\infty} w^2(x)dx < \infty; \\ \lim_{x \to 0} \frac{1 - w(x)}{|x|^2} = \kappa_2 < \infty. \end{cases}$$

(iii) $M = M(n)$ satisfies

$$n^{\frac{1}{4}}/M + M/n^{\frac{1}{3}} \to 0 \quad \text{as } n \to \infty.$$

(iv) The function $W_n(\lambda) = M W(M\lambda)$ can be expanded as

$$W_n(\lambda) = \frac{1}{2\pi} \sum_{\ell = -M}^{M} w\left(\frac{\ell}{M}\right) e^{-i\ell\lambda}.$$

Throughout this paper we use the following nonparametric spectral estimator

$$\hat{g}_n(\lambda) = \int_{-\pi}^{\pi} W_n(\lambda - \mu) I_n(\mu) d\mu, \tag{3}$$

where $I_n(\mu) = (2\pi n)^{-1} \left|\sum_{t=1}^{n} X_t e^{it\mu}\right|^2$. It is known that

$$E\left\{(\hat{g}_n(\lambda) - g(\lambda))^2\right\} = O\left(\frac{M}{n}\right) + O\left(M^{-4}\right); \tag{4}$$

(e.g., Hannan (1970), Brillinger (1981)). The associated minimum contrast estimator of $\theta(g)$ is defined by a value of θ that satisfies the equation

$$\frac{\partial}{\partial\theta} D(f_\theta; \hat{g}_n) = 0. \tag{5}$$

Under certain additional assumptions, Taniguchi (1987) gave the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta(g))$, and showed that $\hat{\theta}_n$ is first-order asymptotically efficient if $\{X_t\}$ is Gaussian and if $g = f_\theta$. That is,

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, I(\theta)^{-1}), \tag{6}$$

where $I(\theta) = (4\pi)^{-1} \int_{-\pi}^{\pi} \frac{\partial \log f_\theta(\lambda)}{\partial\theta} \frac{\partial \log f_\theta(\lambda)}{\partial\theta'} d\lambda$. Since the function $K(\cdot)$ is only assumed to satisfy (iii) of Assumption 1, we can construct infinitely many first-order asymptotically efficient estimators, and it is desirable to distinguish between them. In order to analyse their difference we discuss higher order asymptotics of $\hat{\theta}_n$ and related quantities in the next section.

3 Higher Order Asymptotics of Fundamental Quantities.

In view of the general asymptotic theory, higher order asymptotics of $\sqrt{n}(\hat{\theta}_n - \theta)$ are described in terms of those of

$$B_n \equiv \sqrt{n} \int_{-\pi}^{\pi} a(\lambda) \{A(\hat{g}_n(\lambda)) - A(g(\lambda))\} d\lambda, \tag{1}$$

where $a(\lambda)$ is a continuous function, and $A(\cdot)$ is a four times continuously differentiable function. Expanding $A(\hat{g}_n(\lambda))$ at $g(\lambda)$ we obtain

$$\begin{aligned} B_n &= \sqrt{n} \int_{-\pi}^{\pi} a(\lambda) A^{(1)}\{g(\lambda)\}\{\hat{g}_n(\lambda) - g(\lambda)\} d\lambda \\ &\quad + \frac{\sqrt{n}}{2} \int_{-\pi}^{\pi} a(\lambda) A^{(2)}\{g(\lambda)\}\{\hat{g}_n(\lambda) - g(\lambda)\}^2 d\lambda \\ &\quad + \sqrt{n} \int_{-\pi}^{\pi} a(\lambda) \frac{1}{3!} A^{(3)}\{g^*(\lambda)\}\{\hat{g}_n(\lambda) - g(\lambda)\}^3 d\lambda \\ &= \beta_1 + \beta_2 + \beta_3, \quad \text{(say)}, \end{aligned} \tag{2}$$

where $\hat{g}_n(\theta) \gtrless g^*(\theta) \gtrless g(\theta)$, and $A^{(j)}(x) = (d^j/dx^j)A(x)$. The following lemma asserts that the third term β_3 is negligible in the second-order sense. We have placed the proofs of the lemmas and theorems of this paper in Section 5.

Lemma 1. Under Assumptions 1 and 2,

$$\beta_3 = o_p(n^{-\frac{1}{2}}).$$

The other terms β_1 and β_2 are evaluated as follows.

Lemma 2. Under Assumptions 1 and 2,

$$\beta_2 = \frac{M}{2\sqrt{n}} \int_{-\pi}^{\pi} a(\theta)A^{(2)}\{g(\theta)\}g(\theta)^2 d\theta \cdot \int_{-1}^{1} \omega^2(x)dx + o_p\left(\frac{1}{\sqrt{n}}\right).$$

Lemma 3. Under Assumptions 1 and 2,

$$\begin{aligned}\beta_1 &= \sqrt{n}\int_{-\pi}^{\pi} a(\theta)A^{(1)}\{g(\theta)\}\{I_n(\theta) - g(\theta)\}d\theta \\ &\quad + \frac{\sqrt{n}}{2M^2}\int_{-\pi}^{\pi} a(\theta)A^{(1)}\{g(\theta)\}g''(\theta)d\theta \cdot \int_{-1}^{1} \rho^2 W(\rho)d\rho + o_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Combining the above results gives the following theorem.

Theorem 1. Under Assumptions 1 and 2, it holds that

$$\begin{aligned}&\sqrt{n}\int_{-\pi}^{\pi} a(\theta)\{A(\hat{g}_n(\theta)) - A(g(\theta))\}d\theta \qquad (3)\\ &= \sqrt{n}\int_{-\pi}^{\pi} a(\theta)A^{(1)}\{g(\theta)\}\{I_n(\theta) - g(\theta)\}d\theta \\ &\quad + \frac{\sqrt{n}}{2M^2}\int_{-\pi}^{\pi} a(\theta)A^{(1)}\{g(\theta)\}g''(\theta)d\theta \cdot \int_{-1}^{1} \rho^2 W(\rho)d\rho \\ &\quad + \frac{M}{2\sqrt{n}}\int_{-\pi}^{\pi} a(\theta)A^{(2)}\{g(\theta)\}g(\theta)^2 d\theta \cdot \int_{-1}^{1} \omega^2(x)dx + o_p\left(\frac{1}{\sqrt{n}}\right) \\ &= F_1(a; A^{(1)}; g) + \frac{\sqrt{n}}{2M^2}F_2(a; A^{(1)}; g) + \frac{M}{2\sqrt{n}}F_3(a; A^{(2)}; g) + o_p\left(\frac{1}{\sqrt{n}}\right), \text{ (say)}.\end{aligned}$$

It is also easy to show the following lemma.

Lemma 4. Under Assumptions 1 and 2,

$$E\int_{-\pi}^{\pi} a(\lambda)\{A(\hat{g}_n(\lambda))\}d\lambda = \int_{-\pi}^{\pi} a(\lambda)\{A(g(\lambda))\}d\lambda + o\left(n^{-\frac{1}{2}}\right). \quad (4)$$

4 Higher Order Asymptotic Theory for Minimum Contrast Estimators

In this section we discuss higher order asymptotic efficiency of the minimum contrast estimator $\hat{\theta}_n = (\hat{\theta}_n^1, \ldots, \hat{\theta}_n^p)'$ defined by (5).

Throughout this section we assume $g = f_\theta$, and we do not pursue the validity of the Edgeworth expansion and stochastic expansion for $\hat{\theta}_n$ because it is very technical and is essentially parallel to the discussion given in Section 4.2 of Taniguchi and Kakizawa (2000). Hence we omit the proofs of Lemmas 5, 6, and Theorems 2 and 3 (i.e. in Section 5 we only give the proofs of Lemmas 1–3 and 7 in Section 5).

Lemma 5. Let ξ be an arbitrary fixed number such that $0 < \xi < 3/8$, and let C be a compact subset of Θ. Then, there exists a statistic $\hat{\theta}_n$ which solves (5) such that for some $d > 0$,

$$P_\theta^n\left[|\hat{\theta}_n - \theta| < d n^{\xi - \frac{1}{2}}\right] = 1 - o(n^{-\frac{1}{2}}), \quad (1)$$

uniformly for $\theta \in C$.

Expanding the right hand side of

$$0 = -\frac{\partial}{\partial\theta} D(f_\theta, \hat{g}_n)|_{\theta = \hat{\theta}_n} \quad (2)$$

with respect to $\hat{\theta}_n$ at θ, we have

$$\begin{aligned} 0 = & -\partial_i D(f_\theta, \hat{g}_n) - \partial_i \partial_j D(f_\theta, \hat{g}_n)(\hat{\theta}_n^j - \theta^j) \\ & -\frac{1}{2}\partial_i \partial_j \partial_k D(f_\theta, \hat{g}_n)(\hat{\theta}_n^j - \theta^j)(\hat{\theta}_n^k - \theta^k) + o_p(n^{-1}), \end{aligned} \quad (3)$$

where $\partial_i = \partial/\partial\theta^i$ and Einstein's summation convention is used. Since θ is the pseudo true value defined in equation (2.2) it holds that $\partial_i D(f_\theta, g) = 0$

and we have

$$
\begin{aligned}
&-\partial_i D(f_\theta;\hat g_n) + \partial_i D(f_\theta;g) \qquad (4)\\
&= \partial_i\partial_j D(f_\theta;\hat g_n)(\hat\theta_n^j - \theta_j) + \frac{1}{2}\partial_i\partial_j\partial_k D(f_\theta;\hat g_n)(\hat\theta_n^j - \theta^j)(\hat\theta_n^k - \theta^k) + o_p(n^{-1}).
\end{aligned}
$$

Write

$$
\begin{aligned}
Z_i &= \sqrt{n}\{-\partial_i D(f_\theta;\hat g_n) + \partial_i D(f_\theta;g)\},\\
Z_{ij} &= \sqrt{n}\{\partial_i\partial_j D(f_\theta;\hat g_n) - E\,\partial_i\partial_j D(f_\theta;\hat g_n)\},\\
\bar I_{ij} = \bar I_{ij}(\theta) &= E\{\partial_i\partial_j D(f_\theta;\hat g_n)\},\\
\bar J_{ijk} = \bar J_{ijk}(\theta) &= -E\{\partial_i\partial_j\partial_k D(f_\theta;\hat g_n)\}.
\end{aligned}
$$

Henceforth if θ is scalar we drop the suffices of all the fundamental quantities. Then it follows from (4) that

$$
\begin{aligned}
Z_i = {} & \bar I_{ij}\cdot\sqrt{n}(\hat\theta_n^j - \theta^j) - \frac{1}{\sqrt{n}} Z_{ij}\cdot\sqrt{n}(\hat\theta_n^j - \theta^j)\\
& - \frac{1}{2\sqrt{n}}\bar J_{ijk}\cdot\sqrt{n}(\hat\theta_n^j - \theta^j)\sqrt{n}(\hat\theta_n^k - \theta^k) + o_p(n^{-\frac{1}{2}}). \qquad (5)
\end{aligned}
$$

Solving (5) with respect to $\sqrt{n}(\hat\theta_n^i - \theta^i)$ we obtain the following lemma (c.f. Taniguchi and Kakizawa (2000, Section 4.2)).

Lemma 6. Denote by $\bar I^{ij}$ the (i,j)th element of the inverse matrix of $\{\bar I_{ij}\}$. Then,

$$
\begin{aligned}
\sqrt{n}(\hat\theta_n^i - \theta^i) = {} & \bar I^{ii'} Z_{i'}\\
& + \frac{1}{\sqrt{n}}\left\{\bar I^{ii'}\bar I^{j'k'} Z_{i'j'} Z_{k'} + \frac{1}{2}\bar I^{ii'}\bar I^{j'm'}\bar I^{k'l'}\bar J_{i'j'k'} Z_{m'} Z_{l'}\right\}\\
& + o_p(n^{-\frac{1}{2}}), \qquad i = 1,2,\ldots,p. \qquad (6)
\end{aligned}
$$

The following lemma gives explicit expressions for fundamental quantities.

Lemma 7. Let the parametric family $\mathcal{P}$ contain the true spectral density g. Then under $g = f_\theta$,

(i) $\bar I_{ij} = K^{(2)}(1)\int_{-\pi}^{\pi}\partial_i f_\theta(\lambda)\partial_j f_\theta(\lambda)\cdot f_\theta(\lambda)^{-2}d\lambda + o(n^{-\frac{1}{2}})$,

$$\text{(ii)}\ \bar{\Gamma}_{ijk} = -K^{(3)}(1)\int_{-\pi}^{\pi} \partial_i f_\theta(\lambda)\partial_i f_\theta(\lambda)\partial_k f_\theta(\lambda)\cdot f_\theta(\lambda)^{-3}d\lambda$$

$$-K^{(2)}(1)\int_{-\pi}^{\pi}\{\partial_i f_\theta(\lambda)\partial_j\partial_k f_\theta(\lambda) + \partial_j f_\theta(\lambda)\partial_i\partial_k f_\theta(\lambda) + \partial_k f_\theta(\lambda)\partial_i\partial_j f_\theta(\lambda)\}$$

$$\cdot f_\theta(\lambda)^{-2}d\lambda + o(n^{-\frac{1}{2}})$$

$$\text{(iii)}\ Cov\{Z_i, Z_j\} = 4\pi K^{(2)}(1)^2\int_{-\pi}^{\pi}\partial_i f_\theta(\lambda)\partial_j f_\theta(\lambda)\cdot f_\theta(\lambda)^{-2}d\lambda + o(n^{-\frac{1}{2}})$$

$$= 4\pi K^{(2)}(1)\bar{\Gamma}_{ij} + o(n^{-\frac{1}{2}});$$

$$\text{(iv)}\ E(Z_i Z_{jk}) = 4\pi\left\{K^{(2)}(1)K^{(3)}(1) + 2K^{(2)}(1)^2\right\}\int_{-\pi}^{\pi}\frac{\partial_i f_\theta\partial_j f_\theta\partial_k f_\theta}{f_\theta(\lambda)^3}d\lambda$$

$$+4\pi K^{(2)}(1)^2\int_{-\pi}^{\pi}\frac{\partial_i f_\theta\partial_j\partial_k f_\theta}{f_\theta^2}d\lambda + o(1),$$

$$= \bar{J}_{ijk} + o(1);\quad \text{(say)},$$

$$\text{(v)}\ cum\{Z_i; Z_j; Z_k\} = \frac{32\pi^2 K^{(2)}(1)^3}{\sqrt{n}}\int_{-\pi}^{\pi}\frac{\partial_i f_\theta}{f_\theta}\frac{\partial_j f_\theta}{f_\theta}\frac{\partial_k f_\theta}{f_\theta}d\lambda + o(n^{-\frac{1}{2}}),$$

$$= \frac{1}{\sqrt{n}}\bar{K}_{ijk} + o(n^{-\frac{1}{2}});\quad \text{(say)},$$

$$\text{(vi)}\ E(Z_i) = \frac{K^{(2)}(1)}{\sqrt{n}}\int_{-\pi}^{\pi}\frac{\partial_i f_\theta(\lambda)}{f_\theta(\lambda)^2}b_\theta(\lambda)d\lambda$$

$$+\frac{\sqrt{n}}{2M^2}K^{(2)}(1)\int_{-\pi}^{\pi}\frac{\partial_i f_\theta(\lambda)}{f_\theta(\lambda)^2}\cdot\frac{\partial^2}{\partial\lambda^2}f_\theta(\lambda)d\lambda\cdot\int_{-\infty}^{\infty}\rho^2 W(\rho)d\rho$$

$$-\frac{M}{2\sqrt{n}}\left\{K^{(3)}(1) + 4K^{(2)}(1)\right\}\int_{-\pi}^{\pi}\partial_i f_\theta(\lambda)\cdot f_\theta(\lambda)^{-1}d\lambda\cdot\int_{-\infty}^{\infty}\omega^2(x)dx + o(n^{-\frac{1}{2}}),$$

$$= \frac{1}{\sqrt{n}}B_{1i}(\theta) + \frac{\sqrt{n}}{M^2}B_{2i}(\theta) + \frac{M}{\sqrt{n}}B_{3i}(\theta) + o(n^{-\frac{1}{2}});\quad \text{(say)},$$

where $b_\theta(\lambda) = -(2\pi)^{-1}\sum_{j=-\infty}^{\infty}|j|R(j)e^{-ij\lambda}$.

From Lemmas 6 and 7 the asymptotic moments (cumulants) of $\{\sqrt{n}(\hat{\theta}_n^i - \theta^i)\}$ are evaluated as follows:

$$E\{\sqrt{n}(\hat{\theta}_n^i - \theta^i)\} = \bar{\Gamma}^{ii'}\left\{\frac{1}{\sqrt{n}}B_{1i'}(\theta) + \frac{\sqrt{n}}{M^2}B_{2i'}(\theta) + \frac{M}{\sqrt{n}}B_{3i'}(\theta)\right\}$$

$$+\frac{1}{\sqrt{n}}\left[\bar{\Gamma}^{ii'}\bar{\Gamma}^{j'k'}\bar{J}_{k'i'j'} + 2\pi K^{(2)}(1)\bar{\Gamma}^{ii'}\bar{\Gamma}^{j'k'}\bar{\Gamma}_{i'j'k'}\right] - o(n^{-\frac{1}{2}})$$

$$
\begin{aligned}
&= \frac{\sqrt{n}}{M} B_1^i(\theta) + \frac{M}{\sqrt{n}} B_2^i(\theta) + \frac{1}{\sqrt{n}} B_3^i(\theta) + o(n^{-\frac{1}{2}}) \\
&= \frac{1}{\sqrt{n}} c^i_{n,M}(\theta) + o(n^{-\frac{1}{2}}), \quad \text{(say)}. \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}\left\{\sqrt{n}(\hat{\theta}^i_n - \theta^i), \sqrt{n}(\hat{\theta}^j_n - \theta^j)\right\} &= 4\pi K^{(2)}(1)\bar{I}^{ij} + o(n^{-\frac{1}{2}}) \qquad (8) \\
&= c^{ij} + o(n^{-\frac{1}{2}}), \quad \text{(say)}.
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{cum}\left\{\sqrt{n}(\hat{\theta}^i_n - \theta^i), \sqrt{n}(\hat{\theta}^j_n - \theta^j), \sqrt{n}(\hat{\theta}^k_n - \theta^k)\right\} \\
&= \frac{1}{\sqrt{n}}\Big[\bar{I}^{ii'}\bar{I}^{jj'}\bar{I}^{kk'}\bar{K}_{i'j'k'} + 8\pi K^{(2)}(1)\bar{I}^{ii'}\bar{I}^{jj'}\bar{I}^{kk'}(\bar{J}_{i'j'k'} + \bar{J}_{j'k'i'} + \bar{J}_{k'i'j'}) \\
&\quad + 3(4\pi)^2 K^{(2)}(1)^2\, \bar{I}^{ii'}\bar{I}^{jj'}\bar{I}^{kk'}\bar{\Gamma}_{i'j'k'}\Big] + o(n^{-\frac{1}{2}}) \\
&= \frac{1}{\sqrt{n}} c^{ijk} + o(n^{-\frac{1}{2}}), \quad \text{(say)}. \qquad (9)
\end{aligned}
$$

The J-th order cumulants of $\sqrt{n}(\hat{\theta}^i_n - \theta^i)$ satisfy

$$
\mathrm{cum}^{(J)}\left\{\sqrt{n}(\hat{\theta}^{i_1}_n - \theta^{i_1}), \ldots, \sqrt{n}(\hat{\theta}^{i_J}_n - \theta^{i_J})\right\} = O(n^{-J/2+1}) \qquad (10)
$$

for each $J \geq 3$.

From the general Edgeworth expansion formula (e.g., Taniguchi (1991, p. 14), Taniguchi and Kakizawa (2000, p.169)) we get the following theorem.

Theorem 2.

$$
\begin{aligned}
&P\left[\sqrt{n}(\hat{\theta}^1_n - \theta^1) \leq x_1, \ldots, \sqrt{n}(\hat{\theta}^p_n - \theta^p) \leq x_p\right] \\
&= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} N(\mathbf{y} : \Omega)\left[1 + \sum_{i=1}^{p} \frac{1}{\sqrt{n}} c^i_{n,M}(\theta) H_i(\mathbf{y}) \right. \\
&\quad \left. + \frac{1}{6\sqrt{n}} \sum_{i,j,k=1}^{p} c^{ijk} H_{ijk}(\mathbf{y})\right] d\mathbf{y} + o(n^{-\frac{1}{2}}), \qquad (11)
\end{aligned}
$$

where $\mathbf{y} = (y_1, \ldots, y_p)'$, $\Omega = \{c^{ij}\}$,

$$
N(\mathbf{y} : \Omega) = (2\pi)^{-p/2} |\Omega|^{-\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{y}'\Omega^{-1}\mathbf{y});
$$

and

$$H_{j_1\cdots j_s}(\mathbf{y}) = \frac{(-1)^s}{N(\mathbf{y}:\Omega)}\frac{\partial^s}{\partial y_{j_1}\cdots\partial y_{j_s}} N(\mathbf{y}:\Omega).$$

Next we discuss the second-order asymptotic efficiency of $\{\hat\theta_n\}$. For simplicity we assume that the parameter θ is scalar and we drop all the unnecessary indices from the quantities. Let

$$\begin{aligned} I = I(\theta) &\equiv \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial}{\partial\theta}\log f_\theta(\lambda)\right\}^2 d\lambda;\\ J = J(\theta) &\equiv -\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial}{\partial\theta} f_\theta(\lambda)\right\}^3\{f_\theta(\lambda)\}^{-3}d\lambda\\ &\quad+\frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial^2}{\partial\theta^2} f_\theta(\lambda)\right\}\left\{\frac{\partial}{\partial\theta} f_\theta(\lambda)\right\}\{f_\theta(\lambda)\}^{-2}d\lambda;\\ K = K(\theta) &\equiv \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{\partial}{\partial\theta} f_\theta(\lambda)\right\}^3\{f_\theta(\lambda)\}^{-3}d\lambda. \end{aligned}$$

It is easy to see the following correspondences:

$$\bar I_{ij} \longleftrightarrow 4\pi K^{(2)}(1) I;$$

$$\bar J_{ijk} \longleftrightarrow (4\pi)^2 K^{(2)}(1)^2 J + 8\pi^2\{K^{(2)}(1)K^{(3)}(1) + 4K^{(2)}(1)^2\}K$$

$$\bar K_{ijk} \longleftrightarrow 64\pi^3 K^{(2)}(1)^3 K;$$

$$\bar L_{ijk} \longleftrightarrow 4\pi K^{(2)}(1)(-3J - K) - 2\pi\{K^{(3)}(1) + 4K^{(2)}(1)\}K.$$

Then setting $\kappa = K^{(3)}(1)/K^{(2)}(1)$ we observe that

$$\begin{aligned} E\{\sqrt{n}(\hat\theta_n - \theta)\} &= \frac{\sqrt{n}}{M^2}\frac{1}{8\pi I}\int_{-\pi}^{\pi}\frac{\partial f_\theta}{\partial\theta}\cdot\frac{1}{f_\theta^2}\frac{\partial^2}{\partial\lambda^2}f_\theta d\lambda\cdot\int_{-1}^{1}\rho^2 W(\rho)d\rho\\ &\quad-\frac{M}{\sqrt{n}}\frac{(\kappa+4)}{8\pi I}\int_{-\pi}^{\pi}\frac{\partial f_\theta}{\partial\theta}\frac{1}{f_\theta}d\lambda\int_{-1}^{1}\omega^2(x)dx\\ &\quad+\frac{1}{\sqrt{n}}\left[\frac{1}{4\pi I}\int_{-\pi}^{\pi}\frac{\partial f_\theta}{\partial\theta}\frac{1}{f_\theta^2}b_\theta(\lambda)d\lambda - \frac{J+K}{2I^2} + (\kappa+4)\frac{K}{4I^2}\right]\\ &\quad+o(n^{-\frac{1}{2}})\\ &= \frac{1}{\sqrt{n}}c_{n,M}(\theta) + o(n^{-\frac{1}{2}}); \quad \text{(say)} \qquad (12) \end{aligned}$$

$$\mathrm{Var}\{\sqrt{n}(\hat{\theta}_n - \theta)\} = I^{-1} + o(n^{-\frac{1}{2}}); \tag{13}$$

$$\begin{aligned}
&\mathrm{cum}\left\{\sqrt{n}(\hat{\theta}_n - \theta), \sqrt{n}(\hat{\theta}_n - \theta), \sqrt{n}(\hat{\theta}_n - \theta)\right\} \\
&= \frac{1}{\sqrt{n}}\left[\frac{-3J - 2K}{I^3} + \frac{3K}{2I^3}(\kappa + 4)\right] + o(n^{-\frac{1}{2}}); \\
&= \frac{1}{\sqrt{n}} c^{(3)}_{(\theta)} + o(n^{-\frac{1}{2}}); \qquad \text{(say)}.
\end{aligned} \tag{14}$$

It is seen that $\hat{\theta}_n$ is not second-order asymptotically median unbiased (AMU) (c.f., Taniguchi (1991, p.25)). Let

$$\hat{\theta}^*_n = \hat{\theta}_n - \frac{1}{n} c_{n,M}(\hat{\theta}_n) + \frac{I(\hat{\theta}_n)}{6n} c^{(3)}(\hat{\theta}_n): \tag{15}$$

Then we have,

Theorem 3. (i) The estimator $\hat{\theta}^*_n$ is second-order AMU, i.e.,

$$\lim_{n\to\infty} \sqrt{n}\left|P^n_\theta\{\sqrt{n}(\hat{\theta}^*_n - \theta) \le 0\} - \frac{1}{2}\right| = 0: \tag{16}$$

(ii) The second-order asymptotic distribution of $\hat{\theta}^*_n$ is

$$\begin{aligned}
P^n_\theta\left\{\sqrt{n}(\hat{\theta}^*_n - \theta) \le x\right\} &= \Phi\left\{x\sqrt{I}\right\} + \frac{x^2}{6\sqrt{nI}}\left\{3J + 2K + \frac{3K}{2}(\kappa + 4)\right\}\varphi(x\sqrt{I}) + o(n^{-\frac{1}{2}}) \\
&= \mathrm{Edg}_2(x; \kappa) + o(n^{-\frac{1}{2}}); \quad \text{(say)},
\end{aligned} \tag{17}$$

where $\Phi(y) = \int_{-\infty}^{y} \varphi(t)dt$ and $\varphi(t) = (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)$.
(iii) If

$$\kappa \equiv \frac{K^{(3)}(1)}{K^{(2)}(1)} = -4; \tag{18}$$

then $\mathrm{Edg}_2(x; -4)$ becomes the second-order efficient bound distribution. Hence, if $K(\cdot)$ is such that $\kappa = -4$, then $\hat{\theta}^*_n$ is second-order asymptotically efficient in the class of second-order AMU estimators.

The above results are remarkable.

Remark 1. In the usual parametric estimation theory it is known that "first-order efficiency implies second-order efficiency" (e.g., Ghosh (1994, p.57)). However, our results claim that, in semiparametric estimation, "first-order efficiency does not imply second-order efficiency in general". This is a sharp contrast between the usual parametric estimation and semiparametric estimation.

Remark 2. Theorem 3 can be employed to check whether the estimator based on $K(\cdot)$ leads to a second-order efficient estimator. In Section 2 we gave three examples of $K(\cdot)$, which lead to the first-order asymptotically efficient estimator. Here we examine whether they satisfy the condition $\kappa = K^{(3)}(1)/K^{(2)}(1) = -4$ or not. Let us denote

$$K_1(x) = \log x + \frac{1}{x}; \qquad K_2(x) = (\log x)^2;$$
$$K_3^{\alpha}(x) = (x^{\alpha} - 1)^2.$$

Then it is easily seen that
(i) $K_1^{(2)}(1) = 1$ and $K_1^{(3)}(1) = -4$, hence $\kappa = -4$, which entails the estimator constructed by $K_1(\cdot)$ is second-order asymptotically efficient,
(ii) $K_2^{(2)}(1) = 2$ and $K_2^{(3)}(1) = -6$, hence $\kappa = -3$, which entails the estimator constructed by $K_2(\cdot)$ is not second-order asymptotically efficient, and
(iii) $K_3^{\alpha(2)}(1) = 2\alpha^2$ and $K_3^{\alpha(3)}(1) = 2\alpha^2(3\alpha - 3)$, hence $\kappa = 3\alpha - 3$, which entails that the estimator by $K_3^{\alpha}(\cdot)$ is second-order asymptotically efficient if and only if $\alpha = -1/3$.

5 Proofs

In this section we give the proofs of Theorems and Lemmas in the previous sections.

Proof of Lemma 1. First, by Schwarz's inequality we obtain

$$|\beta_3|^2 \le n \left[\int_{-\pi}^{\pi} a(\lambda)^2 \frac{1}{36} A^{(3)}\{g^*(\lambda)\}^2 d\lambda \cdot \int_{-\pi}^{\pi} \{\hat{g}_n(\lambda) - g(\lambda)\}^6 d\lambda \right]. \quad (1)$$

From Theorem 7.4.2 and the proof of Theorem 7.4.4 of Brillinger (1981), it is seen that

$$E\{\hat g_n(\lambda) - g(\lambda)\} = O(M^{-2});$$
$$\mathrm{cum}^{(J)}\{\hat g_n(\lambda) - g(\lambda), \ldots, \hat g_n(\lambda) - g(\lambda)\} = O\left\{\left(\frac{M}{n}\right)^{J-1}\right\}; \quad \text{for } J \geq 2. \tag{2}$$

For general random variables $Y_1, \ldots, Y_r$,

$$E(Y_1 \cdots Y_r) = \sum_{v=\{v_1,\ldots,v_p\}} \mathrm{cum}\{Y_{j_1}; j_1 \in \nu_1\} \cdots \mathrm{cum}\{Y_{j_p}; j_p \in \nu_p\}; \tag{3}$$

where the summation is over $(j_1, \ldots, j_p)$ satisfying $j_1 + \cdots + j_p = r$. From (2) and (3) it follows that

$$E\left[\{\hat g_n(\lambda) - g(\lambda)\}^6\right] = O\left\{\left(\frac{M}{n}\right)^3\right\}; \tag{4}$$

which, together with Fubini's theorem, leads to

$$n \int_{-\pi}^{\pi} \{\hat g_n(\lambda) - g(\lambda)\}^6 d\lambda = O_p\left\{\frac{M^3}{n^2}\right\} = o_p\left(\frac{1}{n}\right). \tag{5}$$

Next we evaluate the following integral in (1):

$$\int_{-\pi}^{\pi} a(\lambda)^2 A^{(3)}\{g^*(\lambda)\}^2 d\lambda. \tag{6}$$

It is known (see Theorem 7.7.3 of Brillinger (1981)) that there exists $L_1 > 0$ such that

$$\limsup_n \sup_\lambda \{n M^{-1} \log M\}^{1/2} |\hat g_n(\lambda) - E\hat g_n(\lambda)| \leq L_1; \quad a.s. \tag{7}$$

Hence, for sufficiently large n, there exists $L_2 > 0$ such that

$$\sup_\lambda \hat g_n(\lambda) \leq L_2 \quad a.s. \tag{8}$$

Since $A(\cdot)$ is four times continuously differentiable we can see that

$$\int_{-\pi}^{\pi} a(\lambda)^2 A^{(3)}\{g^*(\lambda)\} g^2 d\lambda = O_p(1). \tag{9}$$

Recalling (1), (5) and (9), we have $\beta_3 = O_p(n^{-\frac{1}{2}})$. ■

Proof of Lemma 2. Let us denote $H_2(\lambda) \equiv a(\lambda)A^{(2)}\{g(\lambda)\}$ in β_2. Since we can write $H_2(\lambda)$ as $H_2(\lambda) = H_2^+(\lambda) - H_2^-(\lambda)$, where $H_2^+(\lambda)$ and $H_2^-(\lambda)$ are the positive and negative parts of $H_2(\lambda)$, respectively, we may assume, without loss of generality, $H_2(\lambda)$ in β_2 is nonnegative. First, we evaluate $\mathrm{Var}\{\hat{g}_n(\lambda)\}$. Note that $\hat{g}_n(\lambda)$ is written as

$$\hat{g}_n(\lambda) = \frac{1}{2\pi} \sum_{\ell=-M}^{M} w_\ell \hat{R}(\ell) e^{-i\ell\lambda}, \tag{10}$$

where $w_\ell = w(\frac{\ell}{M})$ and $\hat{R}(\ell) = n^{-1}\sum_{t=1}^{n-|\ell|} X_t X_{t+\ell}$, for $\ell \geq 0$, and $\hat{R}(\ell) \equiv \hat{R}(-\ell)$ for $\ell < 0$. Then we have

$$\begin{aligned}
\mathrm{Var}\{\hat{g}_n(\lambda)\} &= \frac{1}{4\pi^2} \sum_{\ell=-M}^{M} \sum_{\ell'=-M}^{M} w_\ell w_{\ell'} \mathrm{Cov}\{\hat{R}(\ell), \hat{R}(\ell')\} e^{-i(\ell-\ell')\lambda} \\
&= \frac{1}{4\pi^2} \sum_{\ell=-M}^{M} \sum_{\ell'=-M}^{M} w_\ell w_{\ell'} \frac{1}{n^2} \sum_{t=1}^{n-|\ell|} \sum_{t'=1}^{n-|\ell'|} \Big\{ R(t'-t)R(t'+\ell'-t-\ell) \\
&\quad + R(t'+\ell'-t)R(t'-t-\ell) \Big\} e^{-i(\ell-\ell')\lambda} \\
&= A_1 + A_2, \quad \text{(say)}.
\end{aligned} \tag{11}$$

It is seen that

$$\begin{aligned}
A_1 &= \frac{1}{4\pi^2} \sum_{\ell=-M}^{M} \sum_{\ell'=-M}^{M} w_\ell w_{\ell'} \frac{1}{n^2} \sum_{t=1}^{n-|\ell|} \sum_{t'=1}^{n-|\ell'|} \int\!\!\int_{-\pi}^{\pi} e^{i(t'-t)(\mu+\rho)} \\
&\quad \times e^{i(\ell'-\ell)(\rho+\lambda)} g(\mu) g(\rho) d\mu\, d\rho.
\end{aligned} \tag{12}$$

We show that the summation $\sum_{t=1}^{n-|\ell|}\sum_{t'=1}^{n-|\ell'|}$ in (12) can be replaced by $\sum_{t=1}^{n}\sum_{t'=1}^{n}$. For this, observe

$$A_1^\# \equiv \Bigg| \frac{1}{4\pi^2} \sum_{\ell=-M}^{M} \sum_{\ell'=-M}^{M} w_\ell w_{\ell'} \frac{1}{n^2} \sum_{t=n-M}^{n} \sum_{t'=n-M}^{n} \int\!\!\int_{-\pi}^{\pi} e^{i(t'-t)(\mu+\rho)} \cdots$$

$$\times E e^{i(\ell'-\ell)(\lambda+\omega)} g(\mu) g(\lambda) d\mu d\lambda$$

$$= \frac{1}{n^2} \sum_{t=n-M} \sum_{t'=n-M} \int\!\!\int_{-\pi}^{\pi} e^{i(t'-t)(\mu+\lambda)} [M W\{M(\lambda+\omega)\}]^2 g(\mu) g(\lambda) d\mu d\lambda$$

(recall (iv) of Assumption 2)

$$= \frac{M}{n^2} \sum_{t=0} \sum_{t'=0} \int_{-\pi} \int_{-1} d\mu d\lambda' e^{i(t'-t)(\mu+\frac{\lambda'}{M}-\omega)} W^2(\lambda') \times g(\mu) g\left(-\omega+\frac{\lambda'}{M}\right) + \text{lower order terms} \quad (13)$$

(by transformation $\lambda' = M(\lambda+\omega)$)

$$= \frac{M}{n^2} \sum_{q=-M} (M-|q|) \int_{-\pi} e^{iq(\mu-\omega)} g(\mu) d\mu \times \int_{-1} e^{iq\frac{\lambda'}{M}} W(\lambda') g\left(-\omega+\frac{\lambda'}{M}\right) d\lambda' + \text{lower order terms}$$

$$\leq \frac{M^2}{n^2} \sum_{q=-M} |R(q)| \times \int_{-1} W(\lambda') g\left(-\omega+\frac{\lambda'}{M}\right) d\lambda' + \text{lower order terms}$$

$$= O\left(\frac{M^2}{n^2}\right); \quad \text{(by (i) of Assumption 1).} \quad (14)$$

Hence,

$$A_1 = \frac{1}{4\pi^2} \sum_{\ell=-M} \sum_{\ell'=-M} w_\ell w_{\ell'} \frac{1}{n^2} \sum_{t=1} \sum_{t'=1} \int\!\!\int_{-\pi} e^{i(t'-t)(\mu+\lambda)} \times E e^{i(\ell'-\ell)(\lambda+\omega)} g(\mu) g(\lambda) d\mu d\lambda + O\left(\frac{M^2}{n^2}\right)$$

$$= \frac{1}{n} \int\!\!\int_{-\pi} 2\pi D_n(\mu+\lambda) W_n^2(\lambda+\omega) g(\mu) g(\lambda) d\mu d\lambda + O\left(\frac{M^2}{n^2}\right); \quad (15)$$

where $D_n(\mu') = (2\pi n)^{-1} |\sum_{t=1}^{n} e^{-i\mu' t}|^2$. It is known that

$$\int_{-\pi} D_n(\mu+\lambda) g(\mu) d\mu = g(\lambda) + O\left(\frac{\log n}{n}\right); \quad (16)$$

uniformly in ρ (see Zygmund, A. (1959, p. 91)). Thus

$$
\begin{aligned}
A_1 &= \frac{2\pi}{n}\int_{-\pi}^{\pi} W_n^2(\rho+\theta)\{g(\rho) + O(n^{-1}\log n)\}g(\rho)d\rho + O\left(\frac{M^2}{n^2}\right)\\
&= \frac{2\pi M}{n}\int_{-\infty}^{\infty} W^2(\rho')\left\{g\left(\frac{\rho'}{M}-\theta\right) + O(n^{-1}\log n)\right\} g\left(\frac{\rho'}{M}-\theta\right)d\rho' + O\left(\frac{M^2}{n^2}\right)\\
&\qquad (\text{by } \rho' = M(\rho+\theta))\\
&= \frac{2\pi M}{n}\int_{-\infty}^{\infty} W^2(\rho')d\rho' \cdot g(\theta)^2 + O\left(\frac{M\log n}{n^2}\right) + O\left(\frac{M^2}{n^2}\right) + O\left(\frac{1}{nM}\right)\\
&\qquad (\text{by (i) of Assumption 2})\\
&= \frac{M}{n}\int_{-\infty}^{\infty} w^2(x)dx \times g(\theta)^2 + \frac{1}{\sqrt{n}}\, o\left(\frac{1}{\sqrt{n}}\right). && (17)
\end{aligned}
$$

Next we evaluate A_2 in (11). As in the above, it is seen that

$$
\begin{aligned}
A_2 &= \frac{1}{n^2}\sum_{t=1}^{n-|\ell|}\sum_{t'=1}^{n-|\ell'|}\int_{-\pi}^{\pi} e^{i(t'-t)\mu}W_n(\mu+\theta)g(\mu)d\mu\\
&\quad\times \int_{-\pi}^{\pi} e^{i(t'-t)\rho}W_n(\rho+\theta)g(\rho)d\rho\\
&= \frac{2\pi}{n}\int\int_{-\pi}^{\pi} D_n(\mu+\rho)W_n(\mu+\theta)W_n(\rho+\theta)g(\mu)g(\rho)d\mu d\rho + O\left(\frac{M^2}{n^2}\right)\\
&= \frac{2\pi}{n}\int_{-\pi}^{\pi} W_n(\rho-\theta)W_n(\rho+\theta)g(\rho)^2d\rho + O\left(\frac{\log n}{n^2}\right) + O\left(\frac{M^2}{n^2}\right)\\
&= \frac{2\pi M}{n}\int_{-\infty}^{\infty} W(\rho'-2M\theta)W(\rho')g\left(\frac{\rho'}{M}-\theta\right)^2d\rho' + O\left(\frac{\log n}{n^2}\right) + O\left(\frac{M^2}{n^2}\right)\\
&\qquad \text{by } \rho' = M(\rho+\theta). && (18)
\end{aligned}
$$

Since $W(\theta) = (2\pi)^{-1}\int_{-\infty}^{\infty} w(x)e^{-ix\theta}dx$, we can show that

$$
W(\theta) = O(\theta^{-2}); \tag{19}
$$

by integration-by-parts. Hence

$$
A_2 = O\left(\frac{1}{nM}\right) + O\left(\frac{\log n}{n^2}\right) + O\left(\frac{M^2}{n^2}\right) = \frac{1}{\sqrt{n}}\, o\left(\frac{1}{\sqrt{n}}\right). \tag{20}
$$

Recalling (11), (17) and (20) we have

$$\begin{aligned}\sqrt{n}\,E\{\hat g_n(\lambda)-g(\lambda)\}^2 &= \sqrt{n}\left[\mathrm{Var}\{\hat g_n(\lambda)\}+\{E(\hat g_n(\lambda))-g(\lambda)\}^2\right] \\ &= \frac{M}{\sqrt{n}}\int_{-1}^{1} w^2(x)dx \times g(\lambda)^2 + o\left(\frac{1}{\sqrt{n}}\right); \qquad (21)\end{aligned}$$

which, together with Fubini's theorem, implies

$$\begin{aligned}&\frac{\sqrt{n}}{2}E\int_{-\pi}^{\pi} a(\lambda)A^{(2)}\{g(\lambda)\}\{\hat g_n(\lambda)-g(\lambda)\}^2 d\lambda \\ &= \frac{M}{2\sqrt{n}}\int_{-\pi}^{\pi} a(\lambda)A^{(2)}\{g(\lambda)\}g(\lambda)^2 d\lambda \times \int_{-1}^{1} w^2(x)dx + o\left(\frac{1}{\sqrt{n}}\right). \qquad (22)\end{aligned}$$

Next we evaluate the variance of β_2. Denoting $b(\lambda) \equiv \frac{1}{2}a(\lambda)A^{(2)}\{g(\lambda)\}$, it is seen that

$$\begin{aligned}\mathrm{Var}\,\beta_2 &= \mathrm{cum}\{\beta_2,\beta_2\} \\ &= (\sqrt{n})^2\left(\frac{2\pi}{L}\right)^2 \sum_{j=-[\frac{L}{2}]+1}^{[\frac{L}{2}]}\sum_{k=-[\frac{L}{2}]+1}^{[\frac{L}{2}]} b\left(\frac{2\pi j}{L}\right) b\left(\frac{2\pi k}{L}\right) \\ &\quad \times \mathrm{cum}\left[Z(j)^2, Z(k)^2\right] + \text{lower order}, \qquad (23)\end{aligned}$$

where $L = n/M$, and $Z(j) = \hat g_n(\frac{2\pi j}{L}) - g(\frac{2\pi j}{L})$. Here

$$\begin{aligned}\mathrm{cum}[Z(j)^2; Z(k)^2] &= \mathrm{cum}\{Z(j), Z(j), Z(k), Z(k)\} \\ &\quad + \mathrm{cum}\{Z(j)\}\mathrm{cum}\{Z(j), Z(k), Z(k)\} + \cdots \\ &\quad + 2\,\mathrm{cum}^2\{Z(j), Z(k)\} \\ &= C_{jk}^{(4)} + C_{jk}^{(1);(3)} + C_{jk}^{(2);(2)}, \qquad \text{(say)}. \qquad (24)\end{aligned}$$

Hence we can write

$$\begin{aligned}\mathrm{Var}\,\beta_2 &= n\left(\frac{2\pi}{L}\right)^2 \sum_{j=-[\frac{L}{2}]+1}^{[\frac{L}{2}]}\sum_{k=-[\frac{L}{2}]+1}^{[\frac{L}{2}]} b\left(\frac{2\pi j}{L}\right) b\left(\frac{2\pi k}{L}\right) \\ &\quad \times \left\{C_{jk}^{(4)} + C_{jk}^{(1);(3)} + C_{jk}^{(2);(2)}\right\} \qquad (25) \\ &= \gamma_1 + \gamma_2 + \gamma_3, \qquad \text{(say)}.\end{aligned}$$

From (2) it follows that $C^{(4)}_{jk} = O(\frac{M^3}{n^3})$, leading to $\gamma_1 = O(\frac{M^3}{n^2}) = o(\frac{1}{n})$. Similarly it is shown that $\gamma_2 = o(\frac{1}{n})$ and $\gamma_3 = o(\frac{1}{n})$, which imply the assertion. ∎

Proof of Lemma 3. Setting $e(\lambda) \equiv a(\lambda)A^{(1)}\{g(\lambda)\}$, we write β_1 as

$$\begin{aligned}\beta_1 &= \sqrt{n}\int_{-\pi}^{\pi} e(\lambda)\left[\int_{-\pi}^{\pi}\{I_n(\mu) - g(\mu)\}W_n(\lambda-\mu)d\mu\right]d\lambda \\ &\quad + \sqrt{n}\int_{-\pi}^{\pi} e(\lambda)\left[\int_{-\pi}^{\pi} g(\mu)W_n(\lambda-\mu)d\mu - g(\lambda)\right]d\lambda \\ &= L_n^{(1)} + L_n^{(2)}, \qquad \text{(say)}. \end{aligned} \tag{26}$$

Putting $M(\lambda-\mu) = \eta$, we have

$$L_n^{(1)} = \sqrt{n}\int_{-\pi}^{\pi}\left[\int_{M(-\pi-\mu)}^{M(\pi-\mu)} e\left(\mu + \frac{\eta}{M}\right)W(\eta)d\eta\,\{I_n(\mu) - g(\mu)\}\right]d\mu.$$

Letting $J_n = \sqrt{n}\int_{-\pi}^{\pi} e(\mu)\{I_n(\mu) - g(\mu)\}d\mu$, we evaluate

$$|J_n - L_n^{(1)}| = \sqrt{n}\left|\int_{-\pi}^{\pi} A_M(\mu)\{I_n(\mu) - g(\mu)\}d\mu\right|, \tag{27}$$

where

$$A_M(\mu) = \int_{M(-\pi-\mu)}^{M(\pi-\mu)} e\left(\mu + \frac{\eta}{M}\right)W(\eta)d\eta - e(\mu)\int_{-1}^{1} W(\eta)d\eta.$$

For an arbitrary given $\varepsilon > 0$, take $\mu \in [-\pi+\varepsilon, \pi-\varepsilon] \equiv B_\varepsilon$. Since $e(\cdot)$ is continuous, $e(\mu + \frac{\eta}{M})$ is bounded on B_ε. Further, expanding $e(\mu + \frac{\eta}{M})$ as

$$e\left(\mu + \frac{\eta}{M}\right) = e(\mu) + \frac{\eta}{M}e'(\mu) + \frac{1}{2}\left(\frac{\eta}{M}\right)^2 e''(\mu) + \frac{1}{6}\left(\frac{\eta}{M}\right)^3 e'''(\mu^*),$$

and noting that $W(\cdot)$ is an even function, we observe

$$A_M(\mu) = \frac{1}{2M^2}\int_{-1}^{1}\eta^2 W(\eta)d\eta \cdot e''(\mu) + o\left(M^{-3}\right). \tag{28}$$

In the same way as Taniguchi (1987), we get

$$\left|J_n - L_n^{(1)}\right| = O_p\left(\frac{1}{M^2}\right) = o_p\left(\frac{1}{\sqrt{n}}\right). \tag{29}$$

Next it is shown that

$$
\begin{aligned}
L_n^{(2)} &= \sqrt{n}\int_{-\pi}^{\pi} e(\lambda)\left[\int_{M(\lambda-\pi)}^{M(\lambda+\pi)} g\left(\lambda-\frac{\rho}{M}\right)W(\rho)d\rho - g(\lambda)\right]d\lambda \\
&= \sqrt{n}\int_{-\pi}^{\pi} e(\lambda)\left[\int_{-1}^{1}\left\{g(\lambda)+\frac{\rho^2}{2M^2}g''(\lambda)\right\}W(\rho)d\rho + O(M^{-4}) - g(\lambda)\right]d\lambda \\
&= \frac{\sqrt{n}}{2M^2}\int_{-\pi}^{\pi} e(\lambda)g''(\lambda)d\lambda\int_{-1}^{1}\rho^2 W(\rho)d\rho + o\left(\frac{1}{\sqrt{n}}\right); \qquad (30)
\end{aligned}
$$

which implies the desired result. ∎

Proof of Lemma 7 (i) and (ii). From Lemma 4 it follows that

$$\bar{\Gamma}_{ij} = E\{\partial_i\partial_j D(f_\theta;\hat{g}_n)\} = \partial_i\partial_j D(f_\theta;f_\theta) + o\left(n^{-\frac{1}{2}}\right). \qquad (31)$$

Note that

$$
\begin{aligned}
\partial_i\partial_j D(f_\theta;g) &= \int_{-\pi}^{\pi} K^{(2)}\left\{\frac{f_\theta}{g}\right\}\frac{\partial_i f_\theta}{g}\frac{\partial_j f_\theta}{g}d\lambda \\
&\quad + \int_{-\pi}^{\pi} K^{(1)}\left\{\frac{f_\theta}{g}\right\}\frac{\partial_i\partial_j f_\theta}{g}d\lambda. \qquad (32)
\end{aligned}
$$

Since $K^{(1)}(1) = 0$, setting $g = f_\theta$ in (32) we obtain (i) from (31). The assertion (ii) follows from the same argument as (i).

(iii) and (iv). From Theorem 1 we observe

$$Z_i = \sqrt{n}K^{(2)}(1)\int_{-\pi}^{\pi}\partial_i f_\theta(\lambda)\cdot f_\theta(\lambda)^{-2}\{I_n(\lambda) - f_\theta(\lambda)\}d\lambda + o_p(1) \qquad (33)$$

$$
\begin{aligned}
Z_{ij} &= \sqrt{n}\int_{-\pi}^{\pi}\Big[(\partial_i f_\theta\partial_j f_\theta)\{K^{(3)}(1)f_\theta^{-3} + 2K^{(2)}(1)f_\theta^{-3}\} + \partial_i\partial_j f_\theta\cdot K^{(2)}(1)f_\theta^{-2}\Big] \\
&\quad \times\{I_n(\lambda) - f_\theta(\lambda)\}d\lambda + o_p(1); \qquad (34)
\end{aligned}
$$

which, together with Lemma A.3.3 of Hosoya and Taniguchi (1982), lead to the results (iii) and (iv).

(v) The proof follows from (33) and the proof of Lemma 2.2.3 of Taniguchi (1991).

(vi) Setting $a(\lambda) = \partial_{-} f_\theta(\lambda)$ and $A\{g(\lambda)\} = -K^{(1)}\left(\frac{f_\theta}{g} - \frac{1}{g}\right)$ in Theorem 1 we obtain

$$
\begin{aligned}
E(Z_{-}) = {} & E\left[K^{(2)}(1)\sqrt{n}\int_{-\pi}^{\pi} \frac{\partial_{-} f_\theta}{f_\theta^2} \{I_n(\lambda) - f_\theta(\lambda)\}\, d\lambda \right] \qquad (35)\\
& + \frac{K^{(2)}(1)\sqrt{n}}{2M^2}\int_{-\pi}^{\pi} \frac{\partial_{-} f_\theta}{f_\theta^2}\, \frac{\partial^2}{\partial\lambda^2} f_\theta(\lambda)\, d\lambda \cdot \int_{-1}^{1} \rho^2 W(\rho)\, d\rho \\
& - \frac{M}{2\sqrt{n}}\int_{-\pi}^{\pi} \frac{\partial_{-} f_\theta}{f_\theta} \{K^{(3)}(1) + 4K^{(2)}(1)\}\, d\lambda \cdot \int_{-1}^{1} w^2(x)\, dx + o\left(n^{-\frac{1}{2}}\right).
\end{aligned}
$$

It is known (Lemma 2.2.2 of Taniguchi (1991)) that $E\{I_n(\lambda)\} = f_\theta(\lambda) + b_\theta(\lambda) + o(n^{-1})$, hence the assertion is proved. ∎

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