Avoiding bias when estimating the consistency and stability of value-added school effects

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Value-added school effects

- In school effectiveness research, multilevel modelling is the de facto approach to estimating value-added school effects

- There is considerable interest in measuring both the consistency of school effects across subject areas
  - The less consistent school effects are across subjects, the more we should look at subject-specific results in school accountability and improvement exercises

  and the stability of school effects across student cohorts
  - The less stable school effects are over time, the less reliable school effects will be as a guide to school choice

Traditional approach
Separate value-added models

- The traditional approach to measuring the consistency and stability of school effects is to fit separate two-level students-within-schools random-intercept models to student achievement scores in each subject or cohort

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + u_j + e_{ij}$$

$$u_j \sim N(0, \sigma_u^2)$$

$$e_{ij} \sim N(0, \sigma_e^2)$$

and to correlate the resulting empirical Bayes predictions (i.e., 'shrinkage' estimates) of the school random effects across different models $\mathbf{Corr}\left(\tilde{u}_{1j}^{\mathrm{EB}}, \tilde{u}_{2j}^{\mathrm{EB}}\right)$

Modern approach
Joint value-added model

- The modern approach is to fit a single joint random-intercept model

$$y_{1ij} = \mathbf{x}'_{1ij}\boldsymbol{\beta}_1 + u_{1j} + e_{1ij}$$

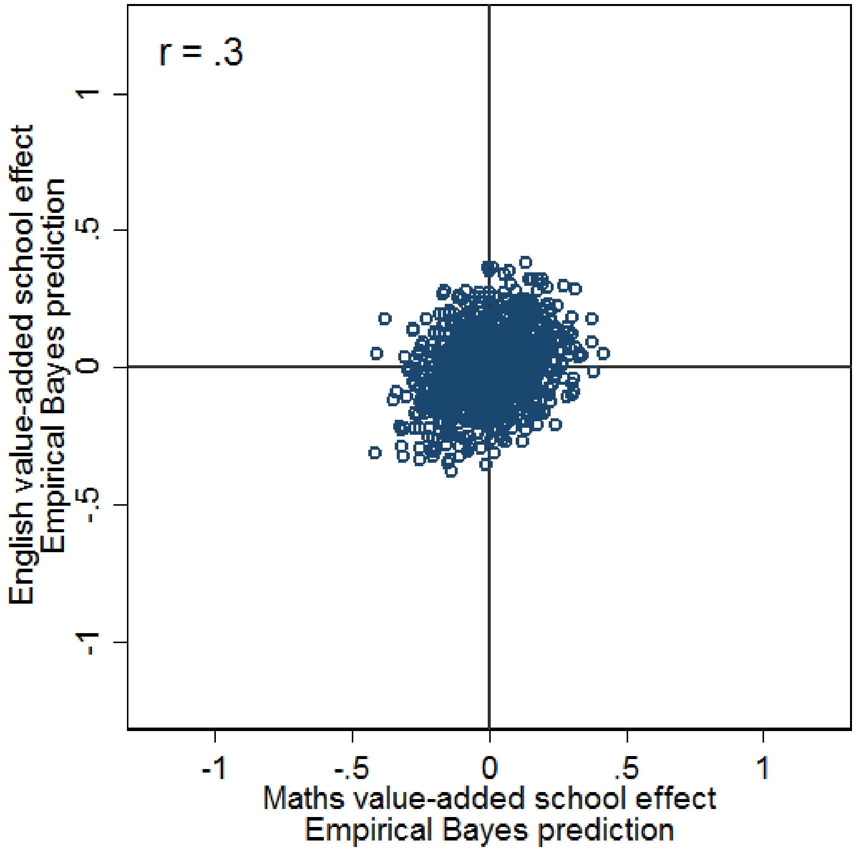
$$y_{2ij} = \mathbf{x}'_{2ij}\boldsymbol{\beta}_2 + u_{2j} + e_{2ij}$$

$$\begin{pmatrix} u_{1j} \\ u_{2j} \end{pmatrix} \sim N\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{u1} & \\ \sigma_{u12} & \sigma^2_{u2} \end{pmatrix}\right\}$$

$$\begin{pmatrix} e_{1ij} \\ e_{2ij} \end{pmatrix} \sim N\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{e1} & \\ \sigma_{e12} & \sigma^2_{e2} \end{pmatrix}\right\}$$

and to estimate the correlation between the school random effects as a model parameter $\mathrm{Corr}(u_{1j}, u_{2j}) = \sigma_{u12}\sigma^{-1}_{u1}\sigma^{-1}_{u2}$

Traditional vs. Modern approach
Different answers



- The traditional approach gives a correlation of 0.3 while the modern approach directly estimates a correlation of 0.5!

Traditional approach
Joint value-added model

- First realise that fitting two separate models is equivalent to fitting the following constrained joint model where we assume schools have uncorrelated rather than correlated effects across subjects

$$y_{1ij} = \mathbf{x}'_{1ij}\boldsymbol{\beta}_1 + u_{1j} + e_{1ij}$$

$$y_{2ij} = \mathbf{x}'_{2ij}\boldsymbol{\beta}_2 + u_{2j} + e_{2ij}$$

$$\begin{pmatrix} u_{1j} \\ u_{2j} \end{pmatrix} \sim N\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{u1} & \\ 0 & \sigma^2_{u2} \end{pmatrix}\right\}$$

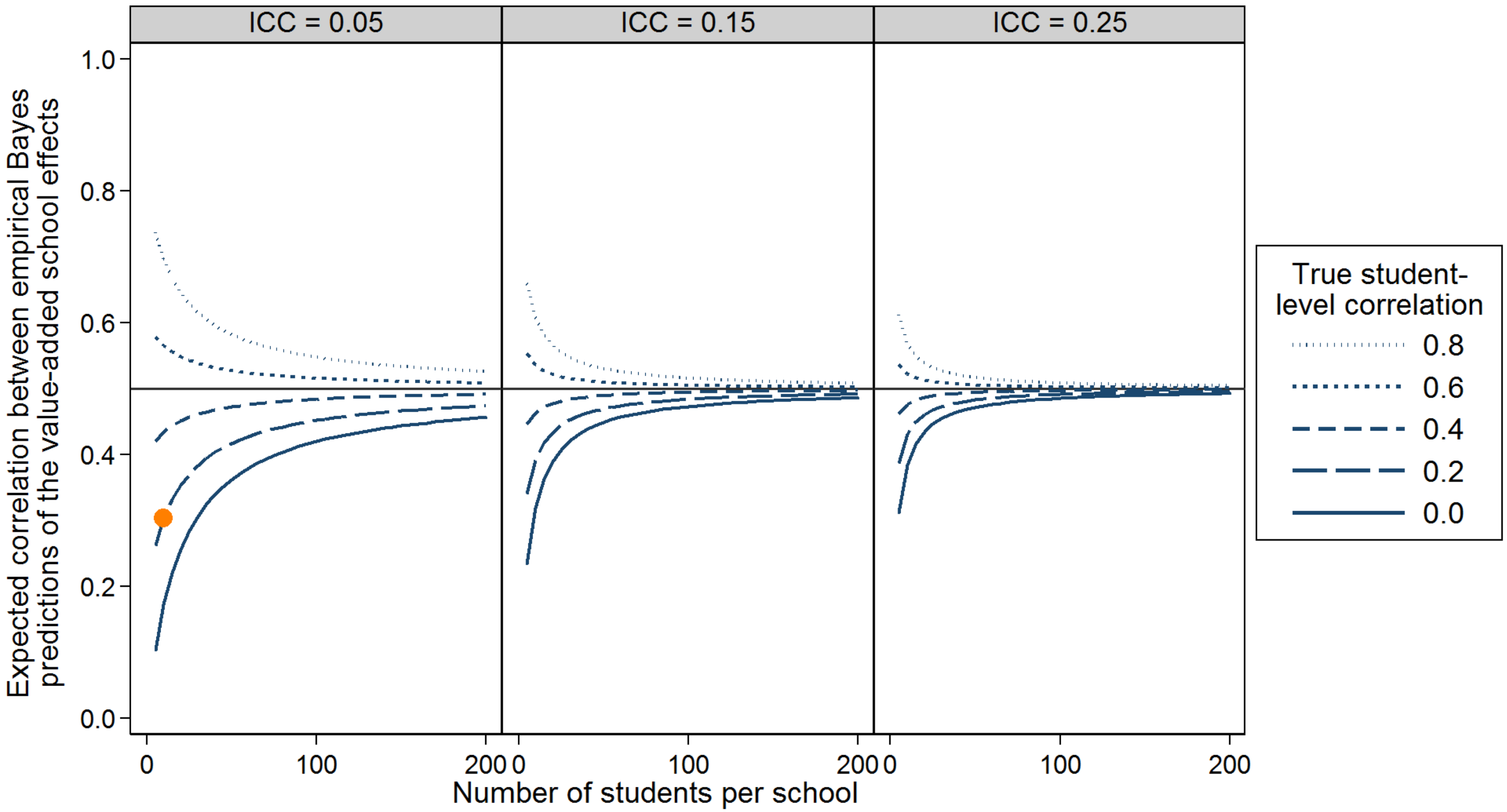
$$\begin{pmatrix} e_{1ij} \\ e_{2ij} \end{pmatrix} \sim N\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{e1} & \\ 0 & \sigma^2_{e2} \end{pmatrix}\right\}$$

Traditional approach
Biased correlation

- Assuming the unconstrained model is the true model and $n_j = n$, it can be shown that the expected correlation between the two sets of school effects from fitting two separate models is

$$\mathrm{Corr}\left(\tilde{u}_{1j}^{\mathrm{EB}}, \tilde{u}_{2j}^{\mathrm{EB}}\right) = \frac{\sigma_{u12} + \frac{\sigma_{e12}}{n}}{\sqrt{\sigma_{u1}^2 + \frac{\sigma_{e1}^2}{n}}\sqrt{\sigma_{u2}^2 + \frac{\sigma_{e2}^2}{n}}}$$

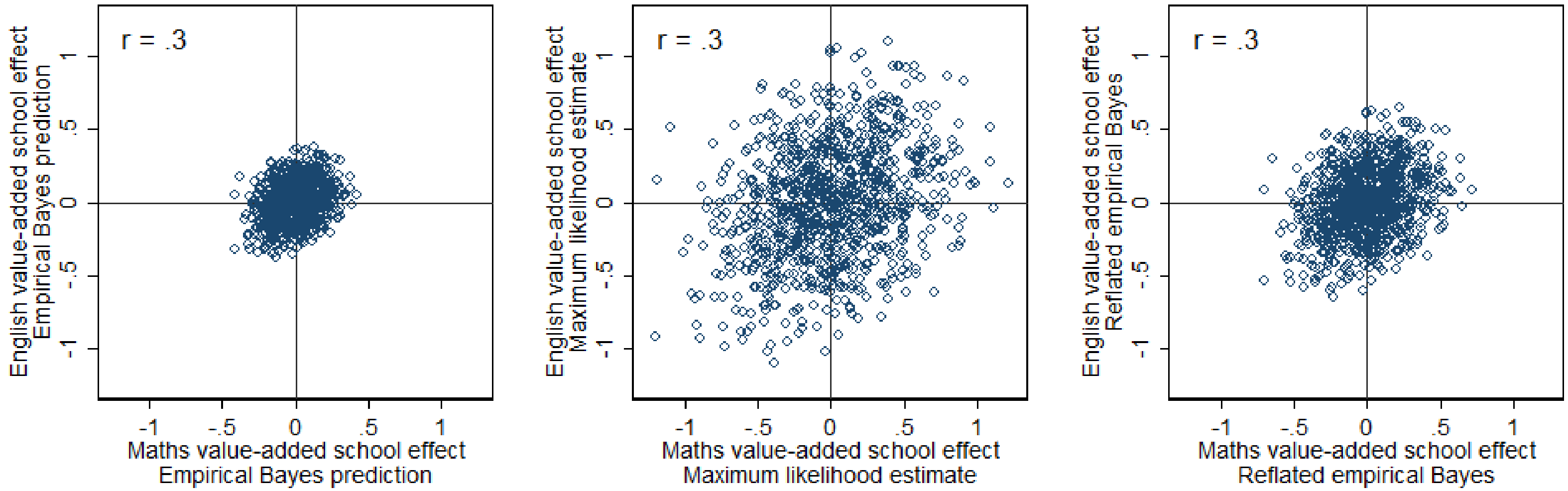
Traditional approach
Biased correlation (cont'd)



- The estimator is pulled towards the student-level correlation; here we under-estimate the consistency correlation

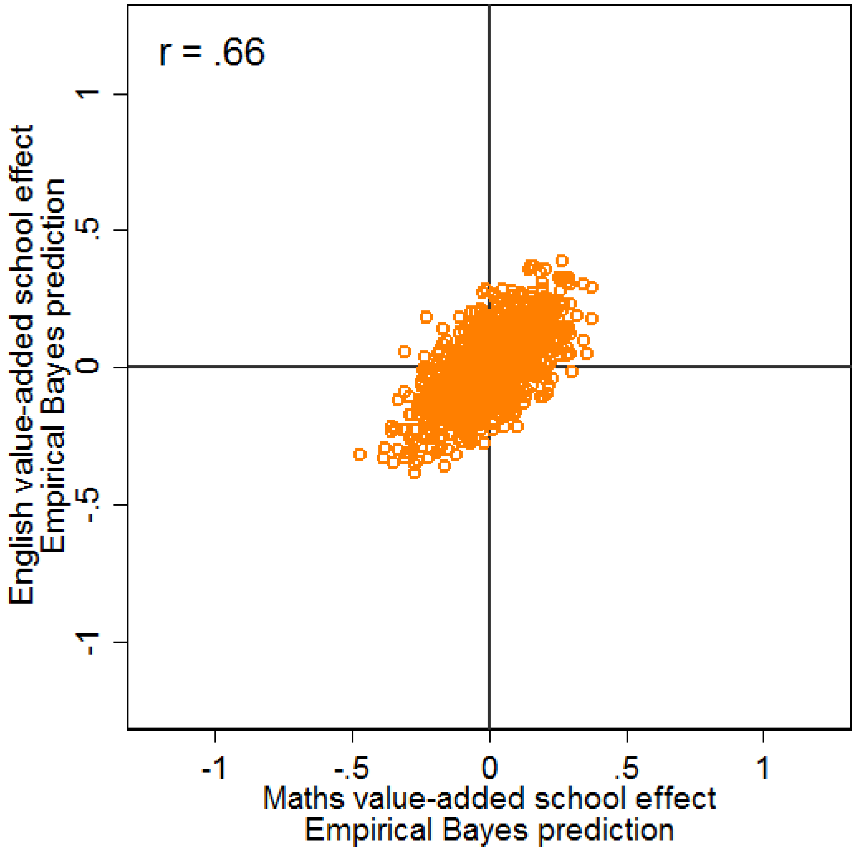
Traditional approach
‘Unshrunken’ or ‘reflated’ effects?

- The bias cannot be avoided by simply correlating ‘unshrunken’ or ‘reflated’ versions of the school effects



- Indeed, while these transformations alter the variances of the English and maths value-added school effects, the traditional consistency correlation remains unchanged, at 0.3

Modern approach (gone wrong)
Different answers



- If we correlate the predicted school effects from the joint model we now get an *upwards* biased correlation!

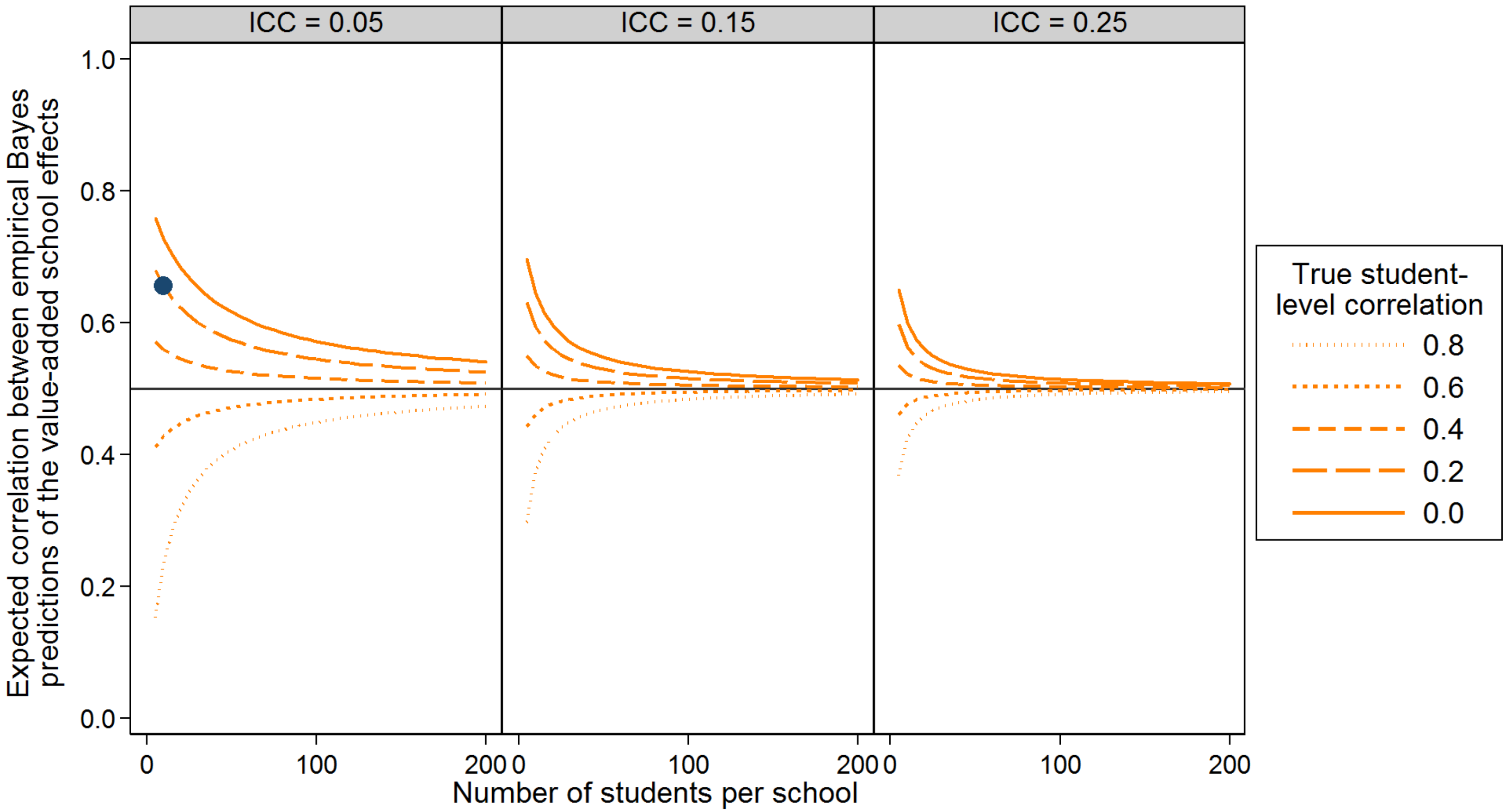
Modern approach (gone wrong)
Biased correlation

- Assuming the unconstrained model is the true model, it can be shown that the expected correlation between the two sets of predicted school effects from fitting this joint model can be derived in the usual way from the following covariance matrix

$$\text{Cov}\begin{pmatrix}\tilde{u}_{1j}^{\text{EB}}\\ \tilde{u}_{2j}^{\text{EB}}\end{pmatrix} = \begin{pmatrix}\sigma_{u1}^2 & \\ \sigma_{u12} & \sigma_{u2}^2\end{pmatrix}\begin{pmatrix}\sigma_{u1}^2+\frac{\sigma_{e1}^2}{n} & \\ \sigma_{u12}+\frac{\sigma_{e12}}{n} & \sigma_{u2}^2+\frac{\sigma_{e2}^2}{n}\end{pmatrix}^{-1}\begin{pmatrix}\sigma_{u1}^2 & \\ \sigma_{u12} & \sigma_{u2}^2\end{pmatrix}$$

- The resulting correlation is again biased, though the form is now more complex

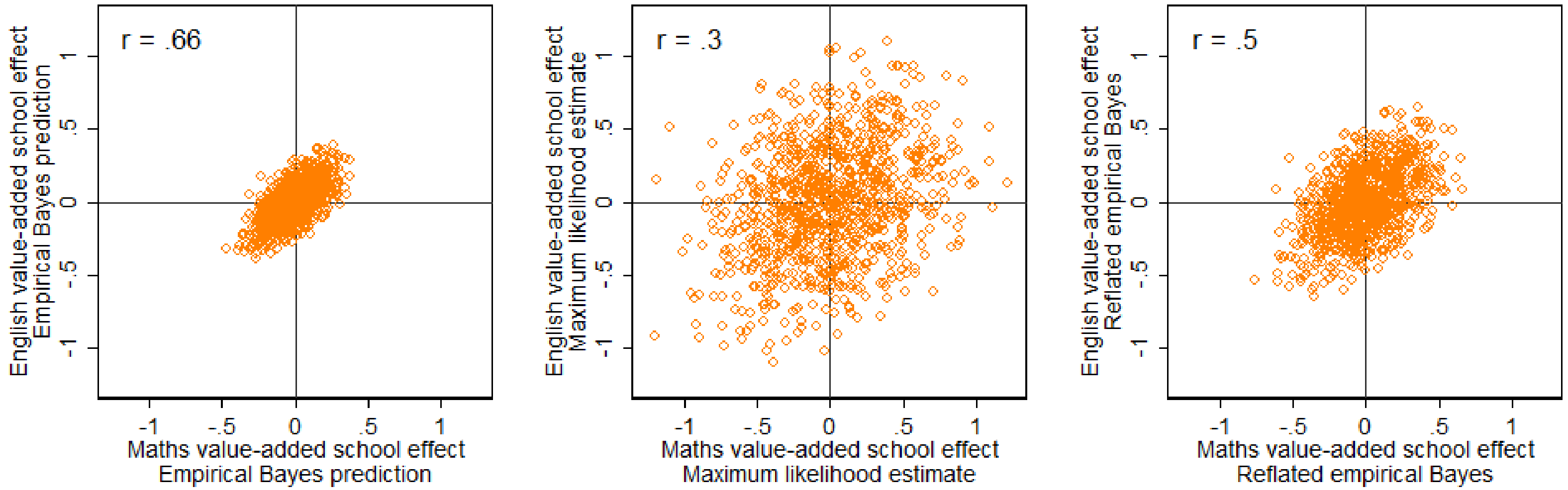
Modern approach (gone wrong)
Biased correlations (cont'd)



- This time the correlation is *pushed away* from the student-level correlation; here we over-estimate the consistency correlation

Modern approach (gone wrong)
‘Unshrunken’ or ‘reflated’ effects?

- This bias also cannot be avoided by simply correlating ‘unshrunken’ versions of these predicted school effects



- However, in contrast to when we fit separate models, this bias can be avoided by correlating ‘reflated’ school effects
- But, there is no need to correlate the predicted school effects at all, as the model-based estimate is unbiased

Conclusion

- The traditional approach to estimating the consistency and stability of school effects gives biased estimates, but these can be avoided by fitting joint models and estimating these correlations directly
- The biases can be substantial when the true consistency and stability correlations differ greatly from the true student residual correlations, especially when school effects are relatively weak and school size relatively small
- Our arguments will apply to other areas of application where researchers interpret correlations between predicted random effects rather than estimating them directly

Article

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Avoiding Bias When Estimating the Consistency and Stability of Value-Added School Effects

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The traditional approach to estimating the consistency of school effects across subject areas and the stability of school effects across time is to fit separate value-added multilevel models to each subject or cohort and to correlate the resulting empirical Bayes predictions. We show that this gives biased correlations and these biases cannot be avoided by simply correlating "unshruken" or "reflated" versions of these predicted random effects. In contrast, we show that fitting a joint value-added multilevel multivariate response model simultaneously to all subjects or cohorts directly gives unbiased estimates of the correlations of interest. There is no need to correlate the resulting empirical Bayes predictions and indeed we show that this should again be avoided as the resulting correlations are also biased. We illustrate our arguments with separate applications to measuring the consistency and stability of school effects in primary and secondary school settings. However, our arguments apply more generally to other areas of application where researchers routinely interpret correlations between predicted random effects rather than estimating and interpreting these correlation directly.

Keywords: *multilevel model; multivariate response; school effects; consistency; stability; value-added*